Harmonic Oscillator Continued

MIT 8.04

Section 14

recall : $\frac{d^2\varphi}{du^2} = (u^2 - \varepsilon)\varphi$

where ~~tx~~ ~~a~~ $au = x$, $a^2 = \frac{\hbar}{m\omega}$

and $\varepsilon = \frac{2E}{\hbar\omega}$ (dimensionless).

this ODE have solution for almost any ε.
so where the quantization of ε arised from?

not every solution can be normalized.

CONSIDER $x \to \infty$. the behavior of ODE.

$$\frac{d^2\varphi}{du^2} = u^2\varphi$$

① can φ be a polynomial?

NO. other with LHS $\sim (n-2)$; RHS $\sim (n+2)$
[if polynomial of order n]. RHS $\gg$ LHS.

② try a solution of the form $\varphi = u^k e^{\frac{\alpha u^2}{2}}$

then RHS $\sim \alpha^2 u^2 \varphi(u)$. RHS $\sim u^2\varphi(u)$; $\alpha = \pm 1$

so this is possible (at lease for the form).

③ but if there is componet of $\alpha = 1$. it diverges.

so. $\varphi(u) = u^k e^{-\frac{u^2}{2}}$ as ~~|x|~~ $\to \infty$.

①②③ $\Rightarrow$ if there is a polynomil-like solution it should be of the form

$$\varphi(u) = h(u)\, e^{-\frac{u^2}{2}}$$

where $h(u)$ is a polynomial.

substitute into dimensionless SE gives

$$\frac{d^2h}{du^2} - 2u\frac{dh}{du} + (\varepsilon - 1)h = 0$$

assume / suppose $h(u) = \sum_{k=0}^{\infty} a_k u^k$

consider the term with u^j in SE.

$$(j+2)(j+1)a_{j+2}u^j - 2j\, a_j u^j + (\varepsilon-1)a_j u^j$$

$$\Rightarrow \sum_{j=0}^{\infty}\left[(j+2)(j+1)a_{j+2} - 2ja_j + (\varepsilon-1)a_j\right]u^j = 0$$

$$\Rightarrow (j+2)(j+1)a_{j+2} = (2j - \varepsilon + 1)a_j$$

[RECURSION RELATION]

$$a_{j+2} = \frac{2j+1-\varepsilon}{(j+2)(j+1)}a_j \qquad j = 0, 1, 2, \cdots$$

initial condition $\Downarrow$ $\boxed{\text{given } h(0),\ h'(0)}$ $\to$

$\boxed{\text{given } a_0}$ $\to a_2, a_4, \cdots a_{2n}$ even solution of $h(u)$

$\boxed{\text{given } a_1}$ $\to a_3, a_5, \cdots a_{2n+1}$ odd solution of $h(u)$

$\}$ full solution $h(u)$

§Quantization of Energy

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recursion relation

$$a_{j+2} = \frac{2j+1-\varepsilon}{(j+2)(j+1)} a_j$$

boundary condition $\left.\begin{matrix} h(0) \\ h'(0) \end{matrix}\right\} \Rightarrow a_0, a_1$

① If $j \to \infty$, a_j doesn't terminate.

then $\frac{a_{j+2}}{a_j} = \lim_{j\to\infty} \frac{2j+1-\varepsilon}{(j+2)(j+1)} = \frac{2}{j}$.

It can be shown, $h(u) \sim e^{u^2}$ as $u \to \infty$.

$\varphi(u)$ diverges at $u \to \infty$

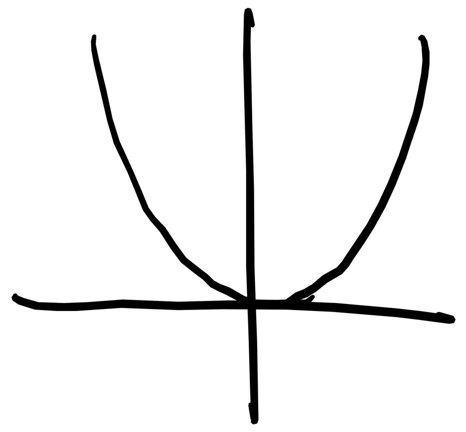
or at least can not be normalized.

② a_j must be a finite series.

Which means ε is an integer

and what's more? $\varepsilon \geq 1$!

▷ $V(x)$ is roughly



, if $\varepsilon = 0$,

then $E = 0$

$E = V_{min}$.

there won't be an eigenstate ◁

$$\varphi(u) = h(u) e^{-\frac{u^2}{2}}$$

where $h(u) = \sum_j a_j x^j$

denote $\varepsilon_j = 2j+1$.

then $h(u) = a_j u^j + a_{j-2} u^{j-2} + \cdots + a_0$

recall

requirement from convergence

$$\varepsilon = \frac{2E}{\hbar\omega} = 2j+1 = 1, 3, \cdots$$

but this requiring BC to be very particular (?).

$\Rightarrow$ let's call j n.

so $\varepsilon_n = 2n+1$

$h_n(u) = a_n x^n + a_{n-2} x^{n-2} + \cdots \left\{\begin{matrix} a_0 \\ a_1 x \end{matrix}\right.$

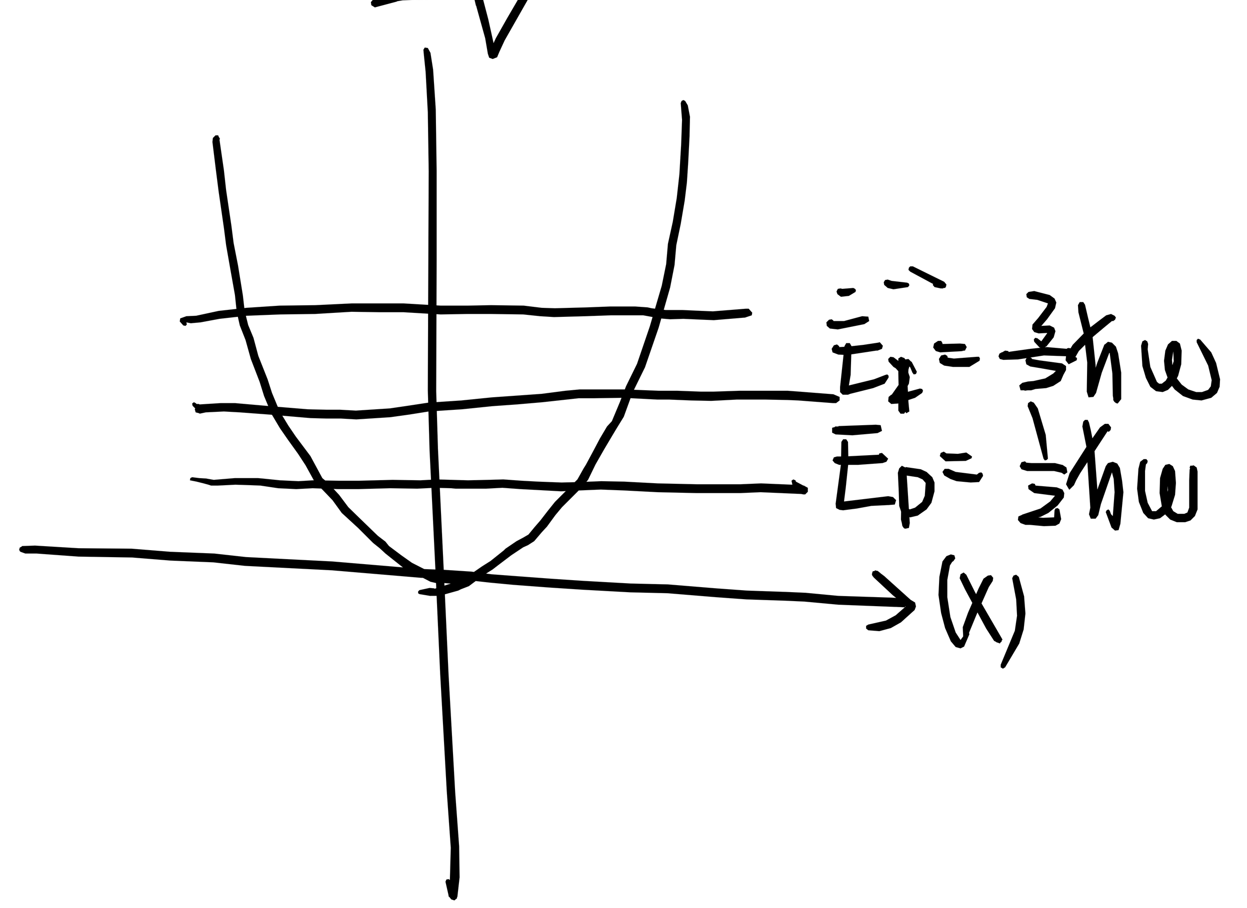
depends on whether n is ODD or EVEN

$$E_n = \frac{\hbar\omega}{2}(2n+1) = \hbar\omega\left(n + \frac{1}{2}\right)$$

$n = 0, 1, 2, \cdots$

compare with the potential

$V(x) = \frac{1}{2} m\omega^2 x^2$



<u>Result</u>: the ~~ene~~ solution for energy eigenstates are either EVEN or ODD

why? : 1-D + symmetric $V(x)$

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§3 Hermite Polynomials

from the [recursion relation]

$$a_{j+2} = \frac{2j+1-\varepsilon}{(j+2)(j+1)} a_j .$$

to make this series finite
we must choose $\varepsilon = 2n+1$ $n = 0, 1, \cdots$
so that the series terminate at $a_n \neq 0$. $a_{n+2} = 0$
then $h(u) = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \cdots \begin{cases} a_0 & (n = \text{even}) \\ a_1 x & (n = \text{ODD}) \end{cases}$

such polynomial that solves SE is called
Hermite Polynomial. denote $H_n(u)$:

$$\frac{d^2 H}{du^2} - 2u \frac{dH}{du} + \cancel{4}(\varepsilon - 1) H = 0$$

$$\Big\downarrow \varepsilon = 2n+1$$

$$\boxed{\frac{d^2 H_n}{du^2} - 2u \frac{dH_n}{du} + \cancel{4}2 n H = 0}$$ Hermite DE

$H_n(u) = 2^n u^n \pm \cdots$

it's arbitrary chosen.
we have the freedom to choose the first coefficient
since it's a LINEAR ODE.

it can be shown the GENERATING FUNCTION of Hermite Polynomial is:

$$e^{-z^2 + 2zu} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(u).$$

then the energy eigenstate solutions are:

$$\varphi_n(x) = N_n H_n\left(x \sqrt{\frac{m\omega}{\hbar}}\right) e^{-\frac{m\omega}{2\hbar} x^2}$$

N_n: normalization constant.

$n = 0, 1, 2, \cdots$