ALGEBRAIC Approach to SHO.

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Section 15

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 = \frac{1}{2} m\omega^2 \left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} \right)$$

search for $\hat{H} = \underbrace{V^{\dagger}V + \text{constant}}$

because $\hat{H} \in$ Hermitian, such construction makes sense

regardless of V is Hermitian or Not.

$$(V^{\dagger}V + \text{constant})^{\dagger} = V^{\dagger}V + \text{constant}^{*}$$

so if constant $\in \mathbb{R}$. this is an Hermitian.

$$\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} \right) \underset{\text{NO}}{\overset{?}{=}} \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right) \left(\hat{x} - \frac{i\hat{p}}{m\omega} \right)$$

$$= \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}[X, P]$$

$[X, P] = i\hbar$

$$= \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} - \frac{\hbar}{m\omega}$$

if $\omega \in \mathbb{R}$. this is a real constant

NOTICE: operators don't necessarily commute

$$\Rightarrow \left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} \right) = \underbrace{\left(\hat{x} + \frac{i\hat{p}}{m\omega} \right)}_{\hat{V}} \underbrace{\left(\hat{x} - \frac{i\hat{p}}{m\omega} \right)}_{\hat{V}^{\dagger}} + \frac{\hbar}{m\omega}$$

so $\hat{H} = \frac{1}{2} m\omega^2 V^{\dagger}V + \frac{1}{2}\hbar\omega$

define $\hat{a} := \sqrt{\frac{m\omega}{2\hbar}}\, V$ (dimensionless)

$\hat{a}^{\dagger} \equiv \sqrt{\frac{m\omega}{2\hbar}}\, V^{\dagger}$

$$[V^{\dagger}, V] = \left[\hat{x} + \frac{i\hat{p}}{m\omega}, \hat{x} - \frac{i\hat{p}}{m\omega} \right] = \frac{2\hbar}{m\omega}$$

Now we have

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right) \qquad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^{\dagger})$$

$$\hat{a}^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i\hat{p}}{m\omega} \right) \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}} (\hat{a}^{\dagger} - \hat{a})$$

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Section 14

$$\hat{H} = \frac{1}{2} m\omega^2 V^+ V + \frac{1}{2}\hbar\omega$$

$$= \underbrace{\hbar\omega}_{[E]}\,\underbrace{(a^+ a + \tfrac{1}{2})}_{\text{dimensionless.}}$$

$$\langle H\rangle_\Psi = \langle\Psi|H|\Psi\rangle$$

$$= \langle\Psi|\,a^+a + \tfrac{1}{2}\,|\Psi\rangle\,\hbar\omega$$

linearity $= [\langle\Psi|a^+a|\Psi\rangle + \langle\Psi|\Psi\rangle\tfrac{1}{2}]\,\hbar\omega$

normalization $= [\langle a^+a\rangle + \tfrac{1}{2}]\,\hbar\omega \geq \tfrac{1}{2}\hbar\omega$ $\Rightarrow$ IF there is any ground state (lower bound).

$(a^+a)^+ = a^+a$

is an Hermitian

~~claim 1 expectation value of Hermitian operator is~~

<u>claim</u> expectation value of a^+a is alway ≥ 0

▷ $\langle a^+a\rangle = \langle\Psi|a^+a|\Psi\rangle = [\langle\Psi|a^+]\,a|\Psi\rangle$

but $\langle\Psi|a^+ = (a|\Psi\rangle)^*$.

so their inner product is actually $|a|\Psi\rangle|^2$

always ~~bigger than zer~~ ≥ 0 ◁

recall

$$[a, a^+] = 1$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right) \qquad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^+)$$

$$\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right) \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\,(\hat{a}^+ - \hat{a})$$

$$\hat{H} = \hbar\omega(a^+a + \tfrac{1}{2})$$

$$E_{\Psi_g} = \frac{\hbar\omega}{2}$$

THEN $a\Psi_g = 0.$

$\Updownarrow$

$$\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\Psi = 0. \rightarrow \left(x + \frac{i}{m\omega}\frac{\hbar}{i}\frac{d}{dx}\right)\Psi = 0.$$

so <u>for ground state</u> : $\left[\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\Psi_g = 0 \quad \text{1st order ODE.}\right]$

rearrange the ODE : $\boxed{\frac{d\Psi_0}{dx} = -\frac{m\omega}{\hbar}x\Psi_0}$

$$\Rightarrow \Psi_0(x) = \underset{\text{normalization coefficient}}{N}\, e^{-\frac{m\omega}{2\hbar}x^2}$$ } perfect Gaussian distribution

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$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})$$

introduce $\hat{N} := \hat{a}^\dagger a$.

$N\psi_0 = 0 \cdot 0$.

$a^\dagger a \psi_0 = 0$

(it can be shown $a\psi_0 = 0$)

claim 1 $[a, a^\dagger] = 1$

from the definity property

since $\hat{a}\psi_0 = 0$.
we call it anihilation operator

claim 2 $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i\hat{p}}{m\omega})$

$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i\hat{p}}{m\omega})$

claim 3 $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)$

$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})$.

always keep in mind
$[a, a^\dagger] \equiv 1$
$[V, V^\dagger] \equiv \frac{2\hbar}{m\omega}$

① check $[\hat{N}, \hat{a}] \equiv [a^\dagger a, a] = [a^\dagger, a]a = -\hat{a}$

$[\hat{N}, \hat{a}^\dagger] \equiv [a^\dagger a, a^\dagger] = a^\dagger[a, a^\dagger] = \hat{a}^\dagger$

there we used $[a, a^\dagger] = 1$

~~theorem~~: identity of COMMUTATOR

$[A, BC] = [A,B]C + B[A,C]$

$[A, BCD] = [A,B]CD + B[A,C]D + BC[A,D]$

$[A, BCDE] = [A,B]CDE + B[A,C]DE + BC[A,D]E + BCE[A,D]$

it's like A walking through the string, ~~the relative posi~~ carrying [ ] by it's side leaving the reluvant ~~position~~ order of B.CDE⋯ unchanged.

② check $[\hat{N}, (\hat{a})^k] = [\hat{N}, \hat{a}]\hat{a}^{k-1} + \hat{a}[\hat{N}, \hat{a}]\hat{a}^{k-2} + \cdots + \hat{a}^{k-1}[\hat{N}, \hat{a}]$

$= -a^{k-1+1} + (-a^{k-1+1}) + \cdots + (-a^{k-1+1})$

$= -k\hat{a}^k$

$[\hat{N}, (a^\dagger)^k] = [\hat{N}, a^\dagger](a^\dagger)^{k-1} + a^\dagger[\hat{N}, a^\dagger](a^\dagger)^{k-2} + \cdots (a^\dagger)^{k-1}[\hat{N}, a^\dagger]$

$= k(\hat{a}^\dagger)^k$

③ check $[a^\dagger, a^k] = [a^\dagger, a]a^{k-1} + a[a^\dagger, a]a^{k-2} + \cdots a^{k-1}[a^\dagger, a]$

$= -k(\hat{a})^{k-1}$

$[a, (a^\dagger)^k] = [a, a^\dagger](a^\dagger)^{k-1} + a^\dagger[a, a^\dagger](a^\dagger)^{k-2} + \cdots (a^\dagger)^{k-1}[a, a^\dagger]$

$= k(a^\dagger)^{k-1}$

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Important Lemma: If $\hat{A}\Psi = 0$

Then $\hat{A}\hat{B}\Psi = [\hat{A},\hat{B}]\Psi$

▷ RHS $= AB\Psi - BA\Psi$
But $A\Psi = 0$, so
$\Rightarrow$ RHS $= AB\Psi$ $=$ LHS ◁

now since we have find ψ_0 and $a\psi_0 = 0$, we try to acting a^+ on ψ_0.

consider the wavefunction $\psi_1 = a^+\psi_0$.

- check if it's an energy eigenstate.

▷ $\hat{N}\psi_1 = \hat{N}a^+\psi_0$
but $\hat{N}\psi_0 = 0$
$\Rightarrow \hat{N}\psi_1 = [\hat{N}, a^+]\psi_0$.

but we have $[\hat{N}, a^+] = a^+$, so $a^+\psi_0 = \psi_1$

i.e. $\hat{N}\psi_1 = \psi_1$, ψ_1 is an eigenvector of $\hat{N}$

thus, an eigenvector of $\hat{H} \Rightarrow$ ~~energy eigenstate~~. ◁

- check if ψ_1 is properly normalized.

▷ $\langle\psi_1|\psi_1\rangle = \langle\psi_0|aa^+|\psi_0\rangle$

but since $a|\psi_0\rangle = 0$.

thus $aa^+|\psi_0\rangle = [a, a^+]|\psi_0\rangle$.

so $\langle\psi_1|\psi_1\rangle = \langle\psi_0|1|\psi_0\rangle \equiv 1$

if ψ_0 is properly normalized ◁

$\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$.

then we consider

$\psi_2 \sim a^+a^+\psi_0$

- check it's energy eigenstate.

$\hat{N}\Psi_2 = \hat{N}(a^+)^2\Psi_0$
but $\hat{N}\Psi_0 = 0$.
$\Rightarrow \hat{N}(a^+)^2\Psi_0 = [\hat{N}, (a^+)^2]\Psi_0$.

but $[\hat{N}, (a^+)^2] = 2(a^+)^2 \Rightarrow \hat{N}\Psi_2 = 2(a^+)^2\psi_0 = 2\Psi_2$.

so $\Psi_2 = (a^+)^2\psi_0$ is an energy eigenstate with $E_2 = (2+\frac{1}{2})\hbar\omega$.

- check if Ψ_2 is normalized.

$\langle\Psi_2|\psi_2\rangle = \langle\psi_0|[(a)^2(a^+)^2|\psi_0\rangle]$

consider $aaa^+a^+\psi_0$

since $a\psi_0 = 0$. $aaa^+a^+\psi_0 = [a, aa^+a^+]\psi_0$
$= a[a, a^+a^+]\psi_0$
$= 2aa^+\psi_0$

but again, since $a\psi_0 = 0$, $2aa^+\psi_0 = 2[a, a^+]\psi_0 = 2\cdot 1\psi_0 = 2\psi_0$

thus $\langle\Psi_2|\Psi_2\rangle = 2\langle\psi_0|\psi_0\rangle = 2$.

- normalization.

easy to see $\psi_2 = \frac{1}{\sqrt{2}}(a^+)^2\psi_0$.

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Important Lemma: If $\hat{A}\Psi = 0$

Then $\hat{A}\hat{B}\Psi = [\hat{A}, \hat{B}]\Psi$

$\triangleright$ $RHS = AB\Psi - BA\Psi$
BUT $A\Psi = 0$, so
$\Rightarrow RHS = AB\Psi = LHS$ $\triangleleft$

now since we have find ψ_0
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but we have $[\hat{N}, a^+] = a^+$, so $a^+\psi_0 = \psi_1$

i.e. $\hat{N}\psi_1 = \psi_1$ ψ_1 is an eigenvector of $\hat{N}$

thus, an eigenvector of $\hat{H} \Rightarrow$ energy eigenstate. $\triangleleft$

- check if ψ_1 is properly normalized.

$\triangleright$ $\langle\psi_1|\psi_1\rangle = \langle\psi_0|aa^+|\psi_0\rangle$

but since $a|\psi_0\rangle = 0$.

thus $aa^+|\psi_0\rangle = [a, a^+]|\psi_0\rangle$.

so $\langle\psi_1|\psi_1\rangle = \langle\psi_0|1|\psi_0\rangle \equiv 1$

if ψ_0 is properly normalized $\triangleleft$

$\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$.

then we consider

$\psi_2 \sim a^+a^+\psi_0$

- check it's energy eigenstate.

$\hat{N}\Psi_2 = \hat{N}(a^+)^2\Psi_0$
but $\hat{N}\Psi_0 = 0$.
$\Rightarrow \hat{N}(a^+)^2\Psi_0 = [\hat{N}, (a^+)^2]\Psi_0$.

but $[\hat{N}, (a^+)^2] = 2(a^+)^2 \Rightarrow \hat{N}\Psi_2 = 2(a^+)^2\psi_0 = 2\Psi_2$.

so $\Psi_2 = (a^+)^2\psi_0$ is an energy eigenstate with $E_2 = (2+\frac{1}{2})\hbar\omega$.

- Check if Ψ_2 is normalized.

$\langle\Psi_2|\psi_2\rangle = \langle\psi_0|\left[(a)^2(a^+)^2|\psi_0\rangle\right]$

consider $aaa^+a^+\psi_0$

since $a\psi_0 = 0$. $aaa^+a^+\psi_0 = [a, aa^+a^+]\psi_0$
$= a[a, a^+a^+]\psi_0$
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but again, since
$a\psi_0 = 0$, $2aa^+\psi_0 = 2[a, a^+]\psi_0 = 2 \cdot 1\psi_0 = 2\psi_0$

thus $\langle\Psi_2|\Psi_2\rangle = 2\langle\psi_0|\psi_0\rangle = 2$.

- normalization.

easy to see $\psi_2 = \frac{1}{\sqrt{2}}(a^+)^2\psi_0$.

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first rewrite $\hat{H} = V^+V + \mathbb{R}$

second dimensionless equation.

$$\hat{H} = \hbar\omega(a^+a + \mathbb{R}).$$

obtaining $\hat{a}$, $\hat{a}^+$ in x, p.

third introduce $N = a^+a$.

verify commutator $[N, a]$, $[N, a^+]$

$[a, a^+]$

$[a, (a^+)^k]$ $[a^+, a^k]$

forth since $N\psi_0 = 0$. solve ψ_0 by ODE.

fifth applying Lemma if $A\psi = 0$

then $AB\psi = [A, B]\psi$

to rewrite ~~ope~~ tensor product by commutator

sixth obtain $(a^+)^k\psi_0$.

and check ~~nor~~ eigenvalue and normalization.

$$\boxed{\text{claim}: \ \psi_n(x) = \underbrace{\frac{1}{\sqrt{n!}}}_{\text{normalization coefficient}} \underbrace{\hat{a}^+\hat{a}^+\cdots\hat{a}^+}_{n}\psi_0(x) \quad \text{and} \quad E_n = \hbar\omega(n+\tfrac{1}{2}).}$$

CREATION and ANIHILATION Operator.

- check claim ①. this is an energy eigenstate

$$\hat{N}\psi_n = \frac{1}{\sqrt{n!}}\hat{N}(\hat{a}^+)^n\psi_0 \underset{\hat{N}\psi_0 = 0}{=} \frac{1}{\sqrt{n!}}\underbrace{[\hat{N}, (\hat{a}^+)^n]}_{= n(\hat{a}^+)^n}\psi_0 = n\psi_n$$

thus

$$\hat{H}\psi_n = \hbar\omega(n+\tfrac{1}{2}).$$

- check claim ②. this is property

$$\langle\psi_n|\psi_n\rangle = \langle\psi_0|(a)^n(a^+)^n|\psi_0\rangle\frac{1}{n!}$$

$$= \langle\psi_0|[a, a^{n-1}(a^+)^n]|\psi_0\rangle\frac{1}{n!}$$

$$= \langle\psi_0|(\hat{a})^{n-1}[a, (a^+)^n]|\psi_0\rangle\frac{1}{n!}$$

$$= \langle\psi_0|\hat{a}^{n-1}n(\hat{a}^+)^{n-1}|\psi_0\rangle\frac{1}{n!}$$

$$= \langle\psi_0|\hat{a}^{n-1}(\hat{a}^+)^{n-1}|\psi_0\rangle\frac{1}{(n-1)!}$$

$$\cdots = 1$$

[NOTICE. ψ_n, ψ_m are eigenstates of Hamilton $\hat{H}$ with ~~id~~ different eigenvalue E_n, E_m. and they are both property normalized, thus.

$$\langle\psi_n|\psi_m\rangle = \delta_{nm}$$]

$$\boxed{\begin{array}{l}\text{we can also show.}\\ \hat{a}\psi_n = \sqrt{n}\,\psi_{n-1}\\ \hat{a}^+\psi_n = \sqrt{n+1}\,\psi_{n+1}\end{array}}$$

that's why we call

$\hat{a} :=$ anihilation operator

$\hat{a}^+ :=$ creation operator.

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☑ expectation values.

keep in mind the anihilation / creation relation

$a\psi_n = \sqrt{n}\,\psi_{n-1}$

$a^+\psi_n = \sqrt{n+1}\,\psi_{n+1}$.

and

$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a+a^+)$

$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(a^+-a)$

$\langle\hat{x}\rangle_{\psi_n} = \langle\psi_n|x|\psi_n\rangle = \left[\int_{-\infty}^{\infty} \underset{\text{EVEN}}{\underset{\uparrow}{|\psi_n|^2}}\, \underset{\text{ODD}}{\underset{\uparrow}{x}}\, dx\right] = 0.$

$\langle\hat{p}\rangle_{\psi_n} = \langle\psi_n|\hat{p}|\psi_n\rangle = 0$

prove? (stationary state of 0 momentum expectation)

• show $\langle\hat{p}\rangle_{\psi_n} = 0$

▷ $\langle\hat{p}\rangle_{\psi_n} \leftrightharpoons \left[\langle\psi_n|\left[(a^+-a)\psi_n\right\rangle\right]$

this gives some $(\,)\psi_{n+1} + (\,)\psi_{n-1}$

but ψ_n are orthogonal to ψ_{n+1} and ψ_{n-1} ◁

☑ uncertainty : find $(\Delta x)_{\psi_n}$.

$(\Delta x)^2_{\psi_n} \equiv \langle\hat{x}^2\rangle_{\psi_n} - \langle\hat{x}\rangle^2_{\psi_n}$

we have proved the second term equal to zero.

but $\langle\hat{x}\rangle_{\psi_n} \equiv \langle\psi|\hat{x}^2|\psi\rangle$

$\hat{x}^2 = (a+a^+)(a+a^+)\frac{\hbar}{2m\omega}$

$= \underbrace{a^2 + a^{+2}}_{\substack{\text{these term are on } \psi_n \\ \mapsto \psi_{n\pm1}\,\cdot\,\langle\psi_n|\psi_{n\pm1}\rangle = 0}} + a^+a + aa^+$.

thus. $(\Delta x)^2_{\psi_n} = \langle\hat{x}^2\rangle_{\psi_n} = \langle\psi_n|(a^+a + aa^+)|\psi_n\rangle\frac{\hbar}{2m\omega}$.

~~but $a^+a = N$ a~~

using the creation/anihilation property

$a^+a|\psi_n\rangle = a^+\sqrt{n}\,\psi_{n-1} = \sqrt{n}\,a^+\psi_{n-1} = \sqrt{n}\sqrt{n}\,\psi_n = n|\psi_n\rangle$

$aa^+|\psi_n\rangle = a\sqrt{n+1}\,\psi_{n+1} = \sqrt{n+1}\,a\psi_{n+1} = (n+1)|\psi_n\rangle$.

thus. $\langle\psi_n|a^+a + aa^+|\psi_n\rangle = (2n+1)\langle\psi_n|\psi_n\rangle = 2n+1$

thus. $(\Delta x)^2_{\psi_n} = \frac{\hbar}{m\omega}(n+\frac{1}{2})$ done !!!