SCATTERING States and STEP Potential

$\underbrace{\text{SCATTERING States}}$

NON-NORMALIZABLE
energy eigen states

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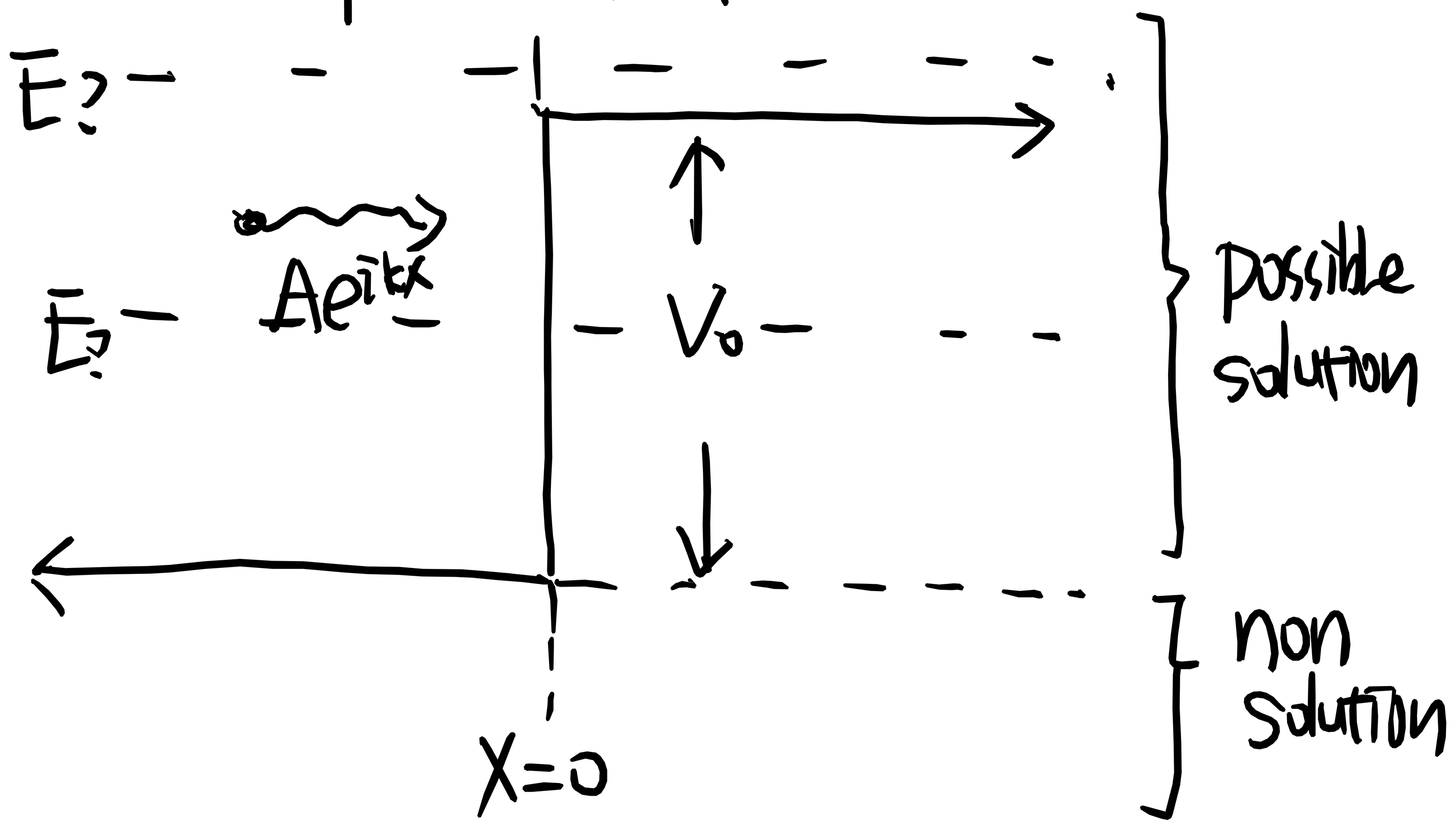
Section
16

§1. Step Potential



§2. Step Potential with $E > V$.

by imagine wavefunction as ordinary wave.
we suggest solution.

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} \\ Ce^{i\bar{k}x} \end{cases}$$

determine coefficient A, B, C (D).
there is no δ in $V(x)$. thus ψ, ψ' continuous.

$$\psi(0_-) = \psi(0_+). \quad \psi'(0_-) = \psi'(0_+).$$

(why is there not a $De^{-i\bar{k}x}$?)

- rigorous derivation of form of $\psi(x)$.

solving from Stationary state's (Time independent Schrödinger equation)

$$\Psi(x,t) = e^{-\frac{iE}{\hbar}t}\psi(x).$$

where $\hat{H}\psi(x) = E\psi(x)$

first consider ~~LH~~ LS of the step.

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + 0. \quad \text{so} \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi(x) \quad (L)$$

while the RS of the step

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0 \quad \text{so} \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E - V_0)\psi(x) \quad (R).$$

- from (L) we see. $\psi'' + \frac{2m}{\hbar^2}E\psi = 0$.

general solution would be. $\psi(x) = Ae^{ikx} + Be^{-ikx}$
where $k = \sqrt{\frac{2mE}{\hbar^2}}$

similarly. (R) gives $\psi'' + \frac{2m}{\hbar^2}(E - V_0)\psi = 0$
general solution would be $\psi(x) = Ce^{i\bar{k}x} + De^{-i\bar{k}x}$
where $\bar{k} = \sqrt{\frac{2m}{\hbar^2}(E - V_0)}$

So the solution do have a tendency to have the form on the ~~RH~~ LHS on this page derived from visualization.

now suppose.

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} \\ Ce^{i\bar{k}x} + De^{-i\bar{k}x} \end{cases} \quad \text{where} \quad k = \sqrt{\frac{2mE}{\hbar}}, \quad \bar{k} = \sqrt{\frac{2m(E-V_0)}{\hbar}}$$

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$$\left.\begin{array}{l} \left.\begin{array}{l} \psi(0_+) = A+B \\ \psi(0_-) = C+D \end{array}\right\} \Rightarrow A+B = C+D \\ \left.\begin{array}{l} \psi'(0_+) = ik(A-B) \\ \psi'(0_-) = i\bar{k}(C-D) \end{array}\right\} \Rightarrow A-B = \frac{\bar{k}}{k}(C-D) \end{array}\right\} \Rightarrow \begin{array}{l} 2A = \frac{\bar{k}+k}{k}C + \frac{k-\bar{k}}{k}D. \\ 2B = \frac{k-\bar{k}}{k}C + \frac{k+\bar{k}}{k}D. \end{array}$$

$$\frac{\bar{k}}{k} = \sqrt{\frac{E-V_0}{E}}$$

forget about D. that's ridiculous.

assume

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} \\ Ce^{i\bar{k}x}. \end{cases}$$

then continuity condition would give

$$\left.\begin{array}{l} \psi_+ = \psi_- \\ \psi'_+ = \psi'_- \end{array}\right\} \Rightarrow \begin{array}{l} A+B = C \\ A-B = \frac{\bar{k}}{k}C. \end{array}$$

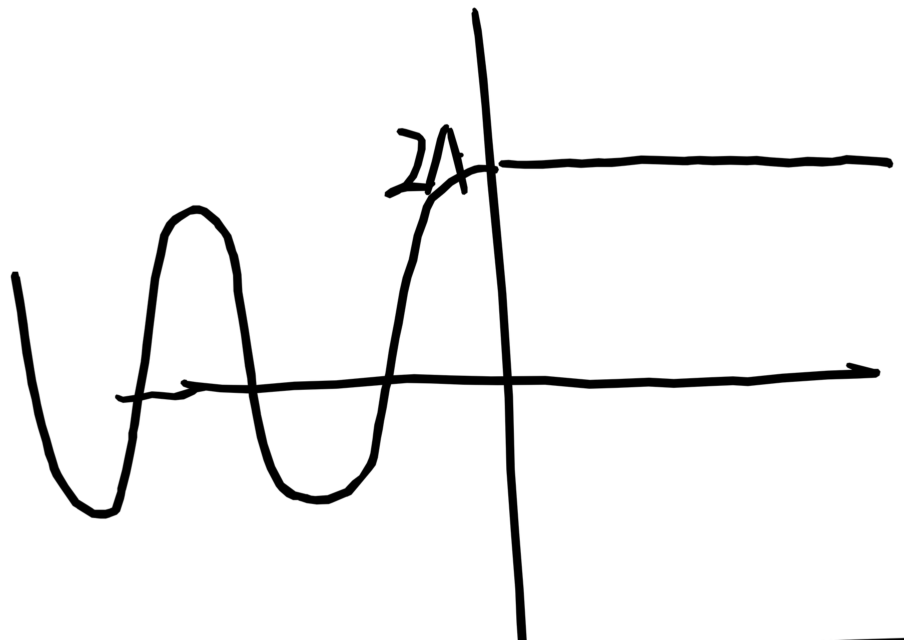
$$\Rightarrow \frac{B}{A} = \frac{k-\bar{k}}{k+\bar{k}} \qquad \frac{C}{A} = \frac{2k}{k+\bar{k}}$$

where $k = \sqrt{\frac{2mE}{\hbar}}$ $\bar{k} = \sqrt{\frac{2m(E-V_0)}{\hbar}}$

- for $E = V_0$. $\bar{k} = 0$.

$$\left.\begin{array}{l} A+B = C \\ A-B = 0 \Rightarrow B = A \end{array}\right\} \Rightarrow C = 2A. \quad \psi(x) = \begin{cases} 2A\cos kx \\ 2A \end{cases}$$

in this particular solution



$$J_L(x) = \frac{\hbar k}{m}(|A|^2 - |A|^2) = 0$$

$$J_R(x) = \cancel{\frac{\hbar k}{m}}|C|^2 = \frac{\hbar\bar{k}}{m}|C|^2 = 0.$$

$$\Downarrow$$

($J_L = J_R$. $\frac{\partial J}{\partial x} = 0$. Satisfies the LOCAL CONSERVATION)

$$\boxed{\text{PROBABILITY CURRENT} \quad J(x) \equiv \frac{\hbar}{m}\mathrm{Im}\left(\psi^* \frac{\partial \psi}{\partial x}\right)}$$

local conservation of probability requires:

$$\frac{\partial J}{\partial x} + \frac{\partial \rho}{\partial t} = 0. \quad \left(\text{for stationary state } \frac{\partial \rho}{\partial t} \equiv 0\right)$$

in general.

$$J_L(x) = J_A - J_B$$

$$J_R(x) = J_C$$

where $J_A = \frac{\hbar k}{m}|A|^2$, $J_B = \frac{\hbar k}{m}|B|^2$.

$$J_C = \frac{\hbar\bar{k}}{m}|C|^2$$

$$\boxed{\begin{aligned} \frac{B}{A} &= \frac{k-\bar{k}}{k+\bar{k}} \\ \frac{C}{A} &= \frac{2k}{k+\bar{k}} \end{aligned}}$$

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probability current:

$$J = \frac{\hbar}{m} \operatorname{Im}\left(\psi^* \frac{\partial\psi}{\partial x}\right). \Rightarrow \begin{cases} J_L = \frac{\hbar k}{m}(|A|^2 - |B|^2). := J_A - J_B \\ J_R = \frac{\hbar \bar{k}}{m} |C|^2 := J_C. \end{cases}$$

by substituting $|A|, |B|, |C|$ with $\square$.

we can prove $J_L = J_R$. so that $\frac{\partial J_{(x=0)}}{\partial x} = 0 = \left(\frac{\partial \rho}{\partial t}\right)_{x=0}$

so current density is every where the same $\Leftarrow$ actually, for stationary state. $\left(\frac{\partial \rho}{\partial t}\right)_{x \in V} \equiv 0$.

<u>define</u> [reflection coefficient] R.

$$R := \frac{J_B}{J_A} \equiv \frac{|B|^2}{|A|^2} = \left(\frac{k-\bar{k}}{k+\bar{k}}\right)^2$$

<u>define</u> [transmission coefficient] T

$$T := \frac{J_C}{J_A} \equiv \frac{\bar{k}|C|^2}{k|A|^2} = \frac{4k\bar{k}}{(k+\bar{k})^2}$$

$\Rightarrow$ <u>check</u> $R + T \equiv 1$ for any $k, \bar{k}$

- for state $E = V_0$

  we have $\bar{k} = 0$, $k = \sqrt{\frac{2mE}{\hbar}}$

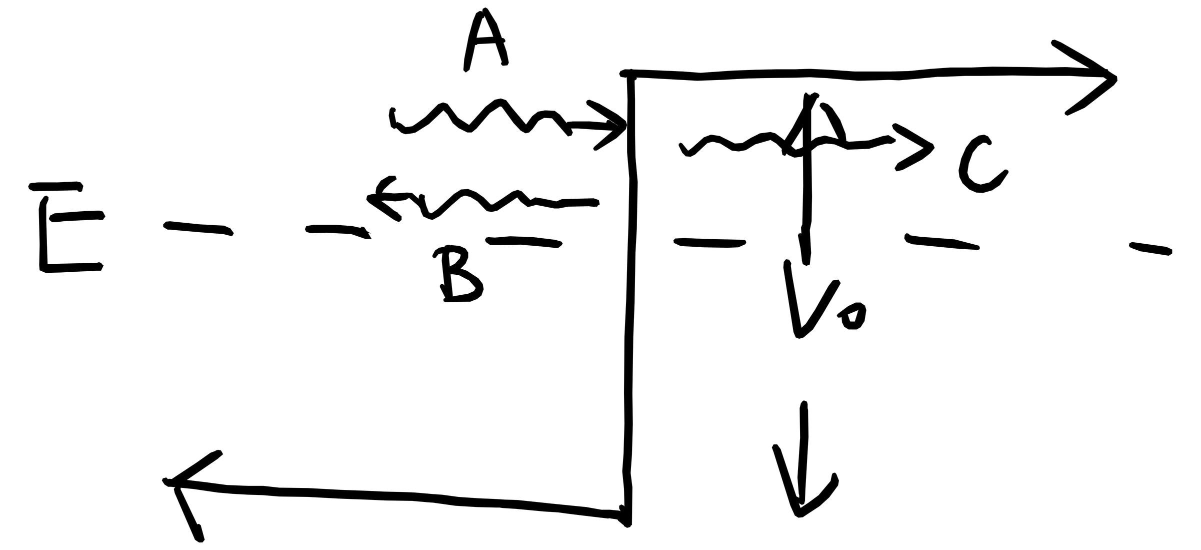
  thus $R_{E=V_0} = \left(\frac{k}{k}\right)^2 = 1$

  $T_{E=V_0} = \frac{0}{(k+\bar{k})^2} = 0$.

ALL "REFLECTED"

NO "transmission"

§3. Step Potential with $E < V_0$



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again, we try to solve SE.

L: $\psi(x) + \frac{2m}{\hbar^2}E\psi = 0$

$\Rightarrow \psi(x) = Ae^{ikx} + Be^{-ikx}$ where $k = \sqrt{\frac{2mE}{\hbar^2}}$

R: $\psi(x) - \frac{2m}{\hbar}(V_0 - E) = 0$.

$\Rightarrow \psi(x) = Ce^{-\kappa x} + \underbrace{De^{\kappa x}}$ where $\kappa = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$

for physical interpretation, let's forget about this term.

again, by applying continuity condition at $x=0$:

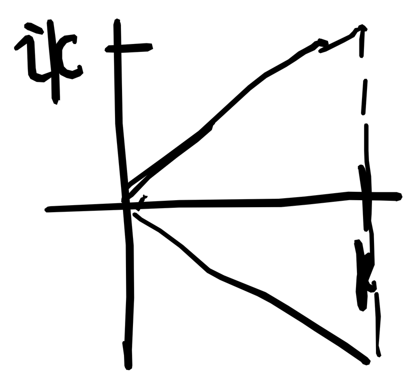
$$\frac{B}{A} = \frac{k - i\kappa}{k + i\kappa} \; ; \; \frac{C}{A} = \frac{2k}{k+i\kappa}$$

since $k, \kappa \in \mathbb{R}$, $\left|\frac{B}{A}\right| = 1$, $\frac{B}{A}$ is merely a phase.

$$\frac{k-i\kappa}{k+i\kappa} = \frac{1}{k^2-\kappa^2}(k-i\kappa)^2 = \frac{1}{k^2-\kappa^2}\left[(k^2-\kappa^2) - 2k\kappa\right]$$

thus, the phase shift is $\tan^{-1}\left(\frac{-2k\kappa}{k^2-\kappa^2}\right)$

a better approach ...



$$\frac{B}{A} = -e^{2\delta(E)i}$$

- then, the probability current

$J_A = \frac{\hbar k}{m}|A|^2$, $J_B = \frac{\hbar k}{m}|B|^2$

$J_C = \frac{i\hbar\kappa}{m}|C|^2$

since we have $\left|\frac{A}{B}\right| = 1$. (due to reality of k and κ).

so $J_A = J_B$, and current conservation requires

$J_C = J_A - J_B \overset{E<V_0}{=\!=} 0$.

- Solution

for $x<0$: $\psi(x) = Ae^{ikx} - Ae^{2i\delta(E)}e^{-ikx}$

for $x>0$: $\psi(x) = Ce^{i\kappa x}$

we can simplify.

$\psi(x) = Ae^{i\delta(E)}\left(e^{-i\delta(E)}e^{ikx} - e^{i\delta(E)}e^{-ikx}\right)$

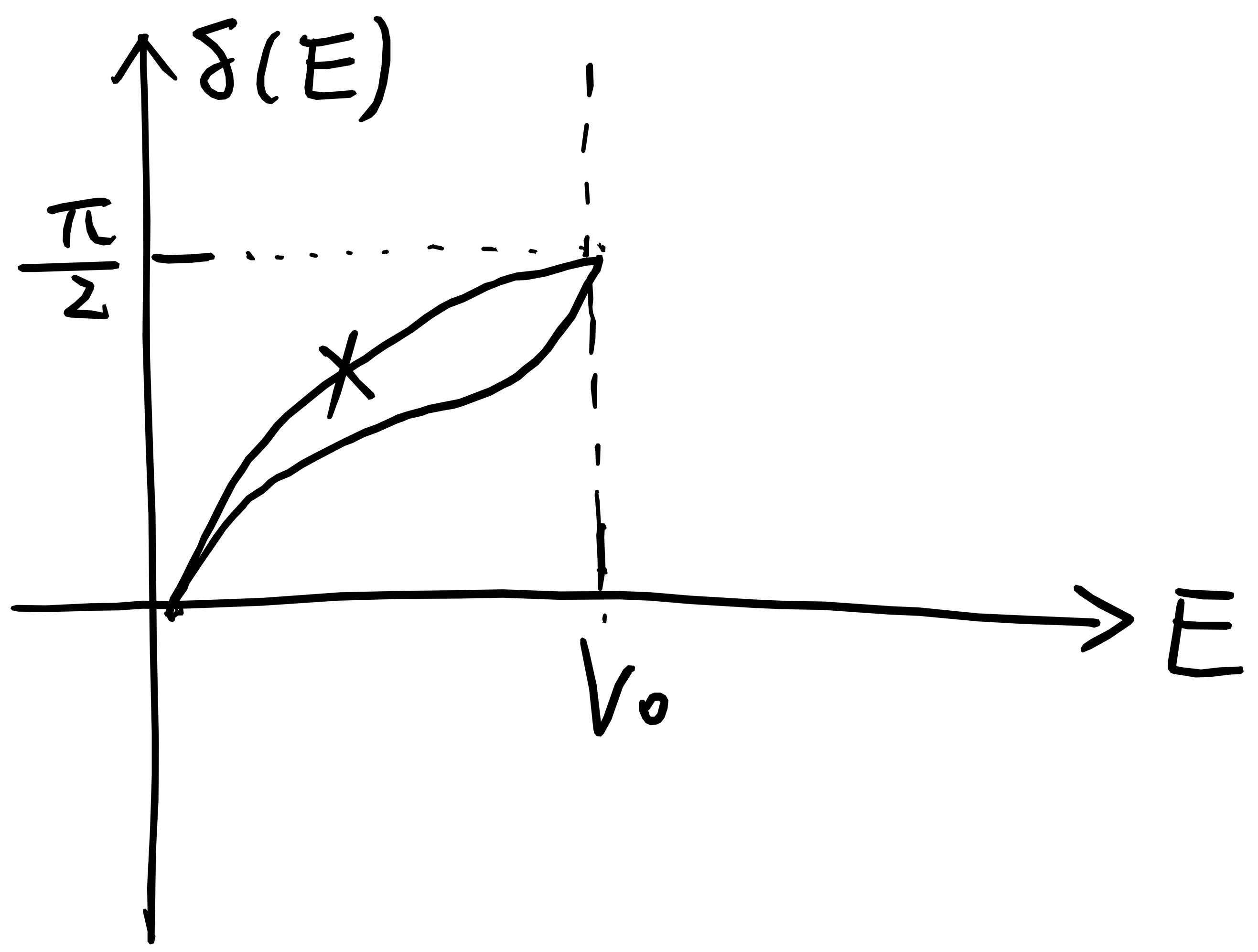
$= 2Ae^{i\delta(E)}i\sin(kx - \delta(E))$ (wavefunction)

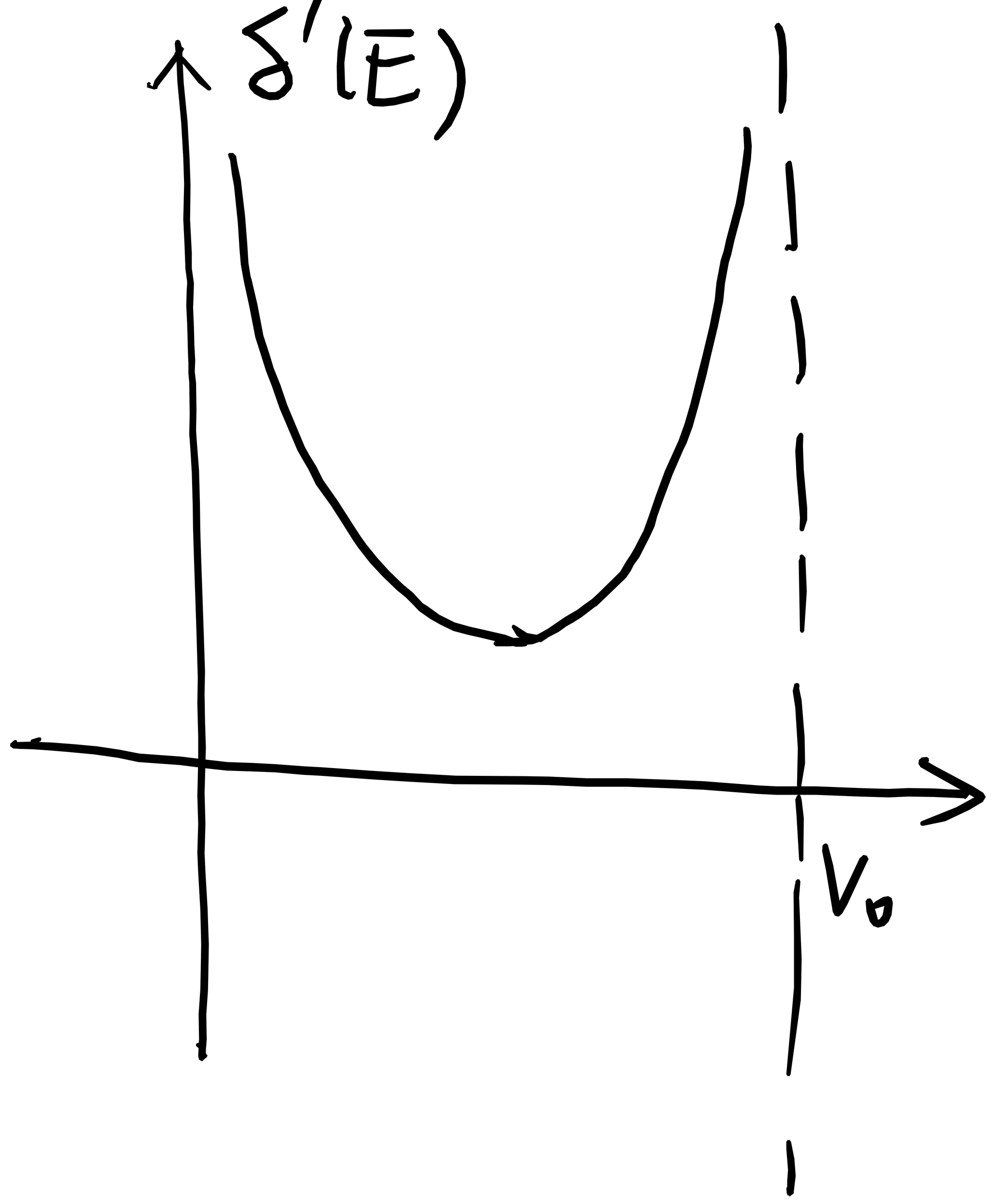
$|\psi(x)|^2 = 4A^2\sin^2(kx - \delta(E))$. (probability density)

[where $\delta(E) = \tan^{-1}\left(\frac{\kappa}{k}\right)$.

$= \tan^{-1}\left(\sqrt{\frac{E}{V_0-E}}\right)$.]

now. since $\delta(E) = \tan^{-1}\left(\sqrt{\frac{E}{V_0 - E}}\right)$.





$$\frac{d\delta(E)}{dE} = \frac{1}{2}\sqrt{\frac{1}{E(V_0 - E)}}$$

§4. WavePackets in the Step Potential.

with $A=1$.

- begin with $E > V_0$

$$\psi(x) = \begin{cases} e^{ikx} + \frac{k-\bar{k}}{k+\bar{k}} e^{-ikx} & x<0 \\ \frac{2k}{k+\bar{k}} e^{i\bar{k}x} \end{cases}$$

solution of TimeIndependent SE.

$$\Rightarrow \Psi(x,t) = \begin{cases} \left(e^{ikx} + \frac{k-\bar{k}}{k+\bar{k}} e^{-ikx}\right) e^{-\frac{iE}{\hbar}t} \\ \frac{2k}{k+\bar{k}} e^{i\bar{k}x} \cdot e^{-\frac{iE}{\hbar}t}. \end{cases}$$

$\Downarrow$

$f(k)\Psi(x,t)$ is still a solution.

notice.
$k = k(E)$
$\bar{k} = \bar{k}(E)$
so if we use k
$\bar{k} = \bar{k}(k)$.
$k > 0$.

$\Downarrow$ superposition

$$\underline{\underline{\Psi(x,t)}} = \int dk\, f(k) \begin{cases} \cdots \\ \cdots \end{cases}$$

WAVE PACKET.

what's more.
we required $E > V_0$
so k has an even
more restrict limit
say $\hat{k}$ such that
$E(\hat{k}) = V_0$

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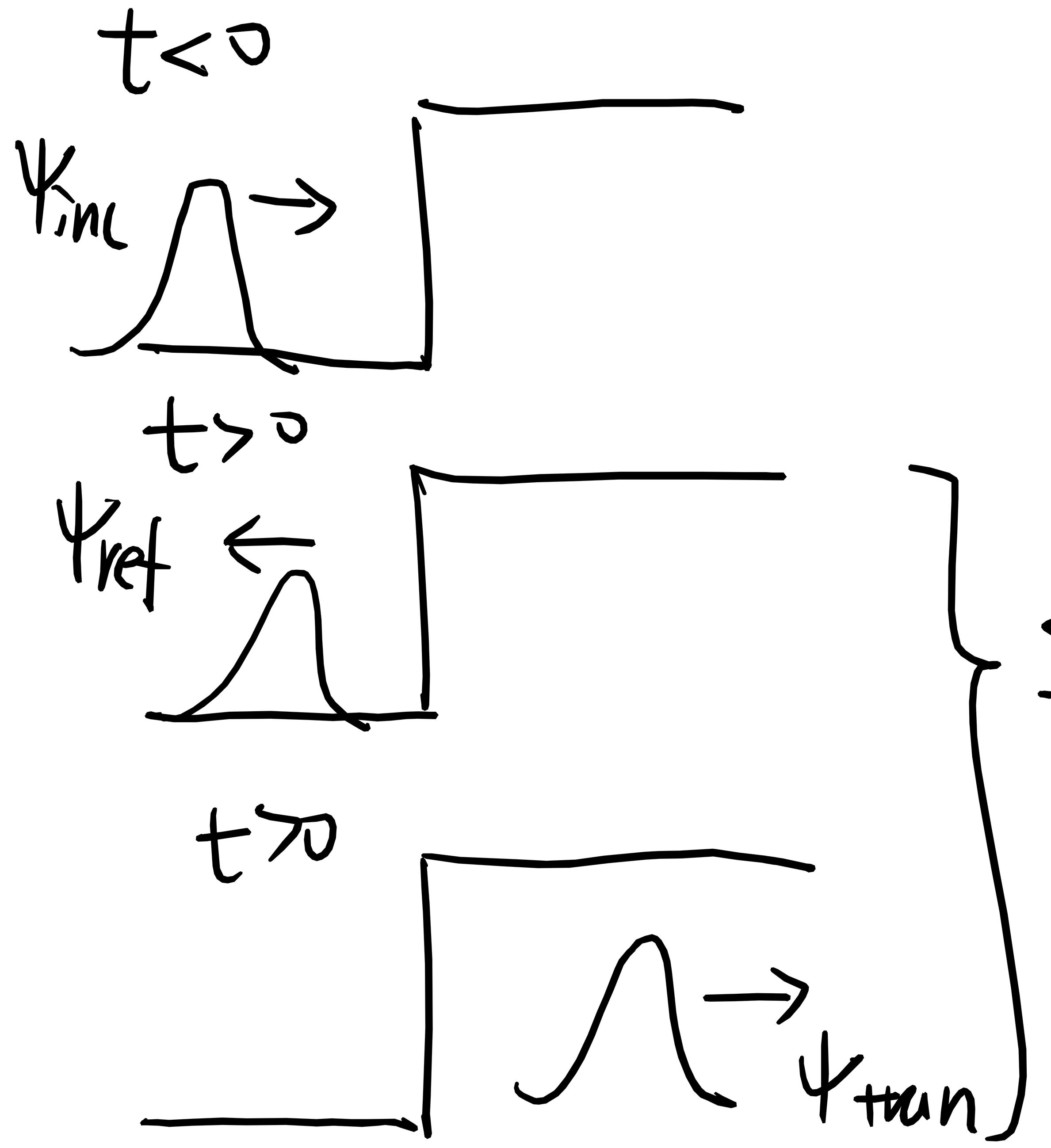
now for $E > V_0$, the wave packet

$$\Psi(x,t) = \int_k^{\infty} dk\, e^{-\frac{iE}{\hbar}t}\, \psi_k(x)$$

since $\psi_k(x) = \begin{cases} & x<0 \\ & x>0 \end{cases}$, we rewrite:

$$\begin{cases} \Psi_{incident}(x,t) = \int_k^{\infty} dk\, f(k)\, e^{ikx} e^{-\frac{iE(k)t}{\hbar}} \\ \Psi_{reflect}(x,t) = \int_{\hat{k}}^{\infty} dk\, f(k)\, e^{-ikx} e^{-\frac{iE}{\hbar}t} \left(\frac{k-\bar{k}}{k+\bar{k}}\right). \\ \Psi_{transmitted}(x,t) = \int_{\hat{k}}^{\infty} dk\, f(k) \left(\frac{2k}{k+\bar{k}}\right) e^{i\bar{k}x} e^{-\frac{iE}{\hbar}t} \end{cases}$$



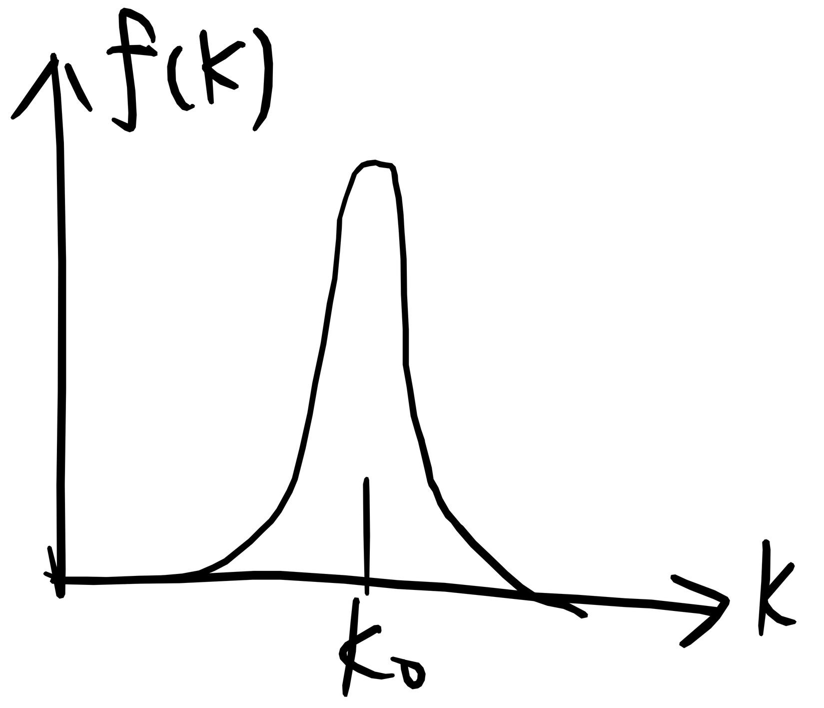
should be $\Longrightarrow$ combined

peaks of wave packet.

$\Leftarrow$

(mainly).

STATIONARY PHASE APPROXIMATION.



then, to make contribution to the integral, the total phase is stationary at $k=k_0$.

or, reversely, the peak of $f(k)$ is at k_0 such that phase is stationary.

i.e. ① $\frac{d}{dt}\left(ikx - \frac{iE(k)t}{\hbar}\right)\Big|_{k=k_0} = 0$

but $E = \frac{\hbar^2 k^2}{2m}$.

ψ_{inc} is only for $x<0$.

$\Rightarrow\ x - \frac{\hbar k_0 t}{m} = 0 \Rightarrow x = \frac{\hbar k_0}{m} t.\quad x<0.$

that's where the incident wave is propagating

② $\frac{d}{dt}\left(-ikx - \frac{iE(k)t}{\hbar}\right)\Big|_{k=k_0} = 0$

so $\Rightarrow\ x + \frac{\hbar k_0 t}{m} = 0 \Rightarrow x = -\frac{\hbar k_0}{m} t \quad x<0.$ i.e. $t>0$.

③ $\frac{d}{dt}\left(i\bar{k}x - \frac{E}{\hbar}t\right)\Big|_{k=k_0} = 0$

...

$\Rightarrow x = \frac{\hbar \bar{k}}{m} t.\quad x>0$ i.e. $t>0$.

(NOTICE: for $t<0$ [stationary phase condition] can not be satisfied anywhere for $\psi_{reflect}$ and ψ_{trans}. so they are RATHER SMALL at this period)

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- move on to $E<V_0$.

$$\Psi_{inc}(x<0,t)=\int_0^k dk\, f(k)e^{ikx}e^{-\frac{iE}{\hbar}t}$$

$$\Psi_{ref}(x<0,t)=\int_0^k dk\, f(k)e^{2i\delta(E)}e^{-ikx}e^{-\frac{iE}{\hbar}t}$$

Ψ_{inc} moves exactly as ~~the~~ in the case $E>V_0$

consider how peak of Ψ_{ref} moves.

phase stationary condition:

$$\left.\frac{d}{dk}\left(2\delta(E)-kx-\frac{E(k)}{\hbar}t\right)\right|_{k=k_0}=0$$

$$\frac{d\delta(E)}{dk}=\frac{d\delta(E)}{dE}\frac{dE}{dk}\ \ldots$$

$$\parallel$$

$$\ldots$$

$$\Rightarrow\ x=-\frac{\hbar k_0}{m}\left(t-2\hbar\delta'(E)\right)$$

but Ψ_{ref} only valid in $x>0$.

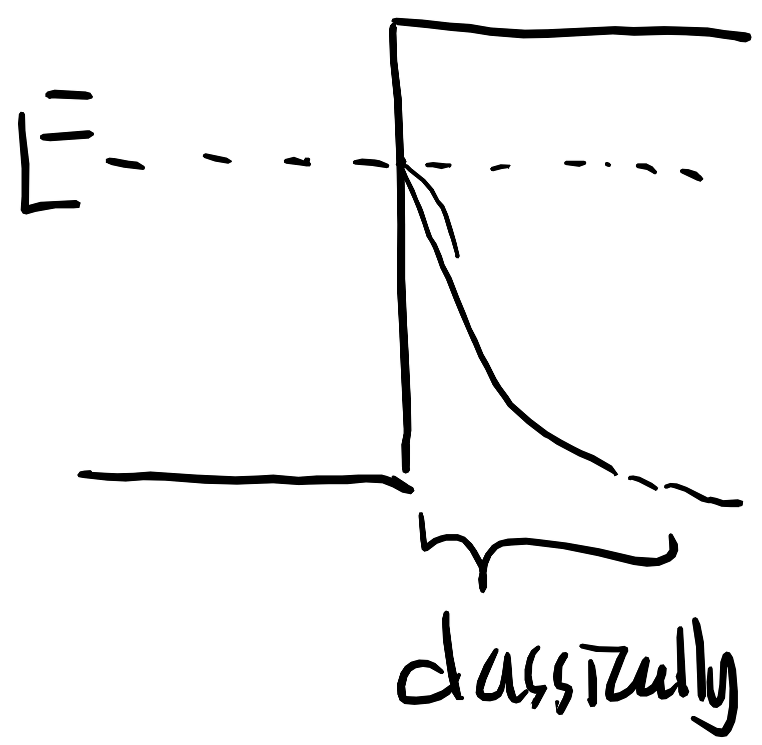
so there is a $\underline{\text{time delay}}$

$$\boxed{\begin{array}{l}\text{only when}\\ t>\text{time delay}\\ \text{the peak exist}\end{array}}$$

$$\begin{pmatrix}\text{time}\\ \text{delay}\end{pmatrix}=2\hbar\delta'(E)$$

$$=2\hbar\left.\left(\frac{d\delta}{dE}\right)\right|_{k=k_0}.$$

Particle on the FORBIDDEN Region



$e^{-\kappa x}$. where $\kappa^2=\frac{2m(V_0-E)}{\hbar^2}$

contradiction if:

① particle in the forbidden region

②' the particle ~~has~~ should have $K.E.>0$.

if ① → ② ✗. if ② → ① ✗

what we do if amend ②' into ②.

② the particle has energy $E<V_0$.

then ①+② indicates the particle has negative kinetic energy

$e^{-\kappa x}$ governs the particle in $x>0$.

so  stretches about $\frac{1}{\kappa}$. beyond that $e^{-\kappa x}$ would be too small.

You need to measure with precision $\Delta x\le\frac{1}{\kappa}$

otherwise you can not tell whether it's in forbidden region or on the left. ~~then~~

but the $\Delta p\geqslant\frac{\hbar}{\Delta x}\geqslant\hbar\kappa$, ⇒ then ΔE would allow $K>0$.