Newtonian Potential & Multipole Expansions

俞千野

武汉大学物理科学与技术学院

2023 年 9 月 28 日

Electric Potential

In electrostatics,Coulomb's law tells us

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{r} \tag{1}$$

The electric field:

$$\vec{E} = \frac{\vec{F}}{q} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} \tag{2}$$

Point charge $\rightarrow$ Continuous charge distribution

$$\vec{E}\left(\vec{r}\right) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}')}{\eta^2}\hat{\eta}d^3\vec{r}' \tag{3}$$

with $\vec{\eta} = \vec{r} - \vec{r}'$,$\hat{\eta} = \vec{\eta}/|\vec{\eta}|$.

Electric Potential

Since $\vec{\nabla} \times \vec{E} = 0$[Appendix I],we can express $\vec{E}$ as $-\vec{\nabla}\varphi$. Then we take the divergence of the both sides of (3) [Appendix II] and get the Poisson's Equation:

$$\vec{\nabla}^2\varphi = -\frac{\rho(\vec{r})}{\epsilon_0} \tag{4}$$

Equation (4) exactly describes the electrostatic field by introducing an electric potential φ(a scalar field).

Newtonian Gravitational Potential

Making some replacements:
$1/4\pi\epsilon_0 \to -G$, $q \to m$, $Q \to M$ and $\varphi \to \phi$
Equation (4) becomes:

$$\vec{\nabla}^2\phi = 4\pi G\rho(t, \vec{r}) \tag{5}$$

Equation (5) exactly describes the Newtonian gravitational field by introducing a gravitational potential ϕ(a 3-dimensional scalar field).

Point mass & Realistic body

Point mass M : $\rho(t,\vec{r}) = M\delta^3(\vec{r})$

$\left[hint: \ \vec{\nabla}^2\left(-\frac{1}{r}\right) = \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})\right]$

$$\phi(\vec{r}) = \phi(r) = -\frac{GM}{r} \tag{6}$$

Realistic body :

· a first approximation :

spherically symmetry

$\Downarrow$ deviations

multipole expansions

(powerful when applied to slight deviations)

Spherical body

The Laplacian operator in spherical polar coordinates(r, θ, φ):

$$\vec{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \tag{7}$$

ρ and ϕ of a spherical body depend on t and r only, and in this case Poisson's equation reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 4\pi G\rho(t, r) \tag{8}$$

Integrating once (the constant of integration was chosen so that the gravitational force at $r = 0$ vanishes)

$$\frac{\partial\phi}{\partial r} = \frac{G}{r^2}\int_0^r \rho(t, r')4\pi r'^2 dr' \tag{9}$$

Spherical body

Define

$$m(t,r) := \int_0^r \rho(t,r')4\pi r'^2 dr' \tag{10}$$

and the body's total mass is

$$M := m(t, r=R) = \int_0^R \rho(t,r')4\pi r'^2 dr' \tag{11}$$

· The potential inside the matter($r < R$):

$$\phi(t,r) = -\frac{GM}{R} - G\int_r^R \frac{m(t,r')}{r'^2}dr' \tag{12}$$

· The potential outside the matter($r > R$):

$$\phi(t,r) = -\frac{GM}{r} \tag{13}$$

Non-spherical body

$$\text{small deviations} \underset{\textit{adopt}}{\overset{\textit{diagnose}}{\leftrightarrows}} \text{multipole expansions}$$

Non-spherical body

We decompose the mass density ρ and Newtonian potential ϕ in spherical harmonics[Appendix IV]

$$\rho(t,\vec{r}) = \sum_{\ell m} \rho_{\ell m}(t,r) Y_{\ell m}(\theta,\varphi) \tag{14}$$

$$\phi(t,\vec{r}) = \sum_{\ell m} \phi_{\ell m}(t,r) Y_{\ell m}(\theta,\varphi) \tag{15}$$

We substitute (14) and (15) into (5) and obtain

$$\mathscr{L}\phi_{\ell m} = 4\pi G r^2 \rho_{\ell m} \tag{16}$$

in which

$$\mathscr{L} := \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \ell(\ell+1) \tag{17}$$

Non-spherical body

With the help of the Green's function method, we find

$$\phi_{\ell m}(t,r) = -\frac{4\pi G}{2\ell+1}\left[r^{\ell}\int_r^{\infty}\frac{\rho_{\ell m}(t,r')}{r'^{\ell+1}}r'^2 dr' + \frac{1}{r^{\ell+1}}\int_0^r r'^{\ell}\rho_{\ell m}(t,r')r'^2 dr'\right] \tag{18}$$

Inserting this into (15), then

$$\phi(t,\vec{r}) = -G\sum_{\ell m}\frac{4\pi}{2\ell+1}\left[q_{\ell m}(t,r)\frac{1}{r^{\ell+1}}Y_{\ell m}(\theta,\varphi) + p_{\ell m}(t,r)r^{\ell}Y_{\ell m}(\theta,\varphi)\right] \tag{19}$$

with

$$q_{\ell m}(t,r) = \int_0^r r'^{\ell}\rho_{\ell m}(t,r')r'^2 dr' \tag{20}$$

$$p_{\ell m}(t,r) = \int_r^R \frac{1}{r'^{\ell+1}}\rho_{\ell m}(t,r')r'^2 dr' \tag{21}$$

Non-spherical body

Outside the matter distribution, where $\rho_{\ell m} = 0$, the term involving $p_{\ell m}$ vanishes

$$\phi_{ext}(t, r, \theta, \varphi) = -G \sum_{\ell m} \frac{4\pi}{2\ell + 1} q_{\ell m}(t, R) \frac{Y_{\ell m}(\theta, \varphi)}{r^{\ell+1}} \tag{22}$$

We've successfully expanded the external potential ϕ_{ext} with $\{Y_{\ell m}(\theta, \varphi)\}$ and $\{\frac{1}{r^{\ell+1}}\}$.

Non-spherical body

We can rewrite (22) in the following form:

$$\phi_{ext} = \sum_{\ell} \phi_{\ell} \tag{23}$$

with

$$\begin{aligned}\phi_{\ell} &= -G \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \left(\int_0^R r^{\ell} \rho_{\ell m}(t,r) r^2 dr \right) \frac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+1}} \\ &= -\sum_{m=-\ell}^{\ell} \frac{4\pi G}{2\ell+1} \left[\int_0^R r^{\ell} \left(\int Y^*_{\ell m}(\theta',\varphi') \rho(t,r,\theta',\varphi') d\Omega' \right) r^2 dr \right] \frac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+1}} \\ &\propto -\frac{GMR^{\ell}}{r^{\ell+1}}\end{aligned} \tag{24}$$

Multipole moments

Multipole moments:

$$\begin{aligned} I_{\ell m}(t) &:= q_{\ell m}(t, R) \\ &= \int_0^R r^\ell \left(\int Y_{\ell m}^*(\theta, \varphi) \rho(t, \vec{r}) \sin\theta d\theta d\varphi \right) r^2 dr \\ &= \int_V r^\ell Y_{\ell m}^*(\theta, \varphi) \rho(t, \vec{r}) d^3\vec{r} \ \left(\propto MR^\ell \right) \end{aligned} \tag{25}$$

so

$$\phi_{ext}(t, r, \theta, \varphi) = -G \sum_{\ell m} \frac{4\pi}{2\ell + 1} I_{\ell m}(t) \frac{Y_{\ell m}(\theta, \varphi)}{r^{\ell+1}} \tag{26}$$

An analogy of (26):

$$\vec{v} = v^i \vec{e}_i \tag{27}$$

Multipole moments

Monopole momnet:

$$I_{00} = \int \rho Y_{00} d^3\vec{x} = \frac{M}{\sqrt{4\pi}} \tag{28}$$

Dipole moments:

$$\begin{aligned} I_{10} &= \sqrt{\frac{3}{4\pi}} \int \rho z d^3\vec{x} \\ I_{1\pm1} &= \mp\sqrt{\frac{3}{8\pi}} \int \rho(x \pm iy) d^3\vec{x} \end{aligned} \tag{29}$$

If we place the origin of the coordinate system at the body's center-of-mass, so that $\int \rho\vec{x} d^3\vec{x} = \vec{0} \Rightarrow I_{10} = I_{1\pm1} = 0$.
(Q: Differences between the mass multipole moments introduced here and the charge multipole moments defined in electromagnetism?)

Multipole moments

Spherically symmetric $\rightarrow$ only I_{00} is non-zero

Axially symmetric about z axis $\rightarrow$ only $I_{\ell 0}$ is non-zero

Axially symmetric body

It is conventional to express the moments in terms of dimensionless quantities J_ℓ defined by

$$J_\ell := -\sqrt{\frac{4\pi}{2\ell+1}}\frac{I_{\ell 0}}{MR^\ell} \tag{30}$$

The gravitational potential of an axially symmetric body can then be written in the form

$$\phi_{ext}(t,\vec{r}) = -\frac{GM}{r}\left[1 - \sum_{\ell=2}^{\infty} J_\ell \left(\frac{R}{r}\right)^\ell P_\ell(\cos\theta)\right] \tag{31}$$

STF vs $\{Y_{\ell m}\}$

Alternative decomposition: Using **tensorial combinations** of the unit vector $\vec{n} := \vec{r}/r$ (instead of spherical harmonics).

Each tensor that we shall construct from $\vec{n}$ will have the property of **being symmetric(S) under the exchange of any two of its indices**, and of **being tracefree(TF) in any pair of indices**; these tensors are known as symmetric tracefree tensors, or STF tensors.

STF vs $\{Y_{\ell m}\}$

The decompositions in STF tensors and spherical harmonics both involve building blocks that consist of irreducible representations of the rotation group labelled by a multipole index ℓ.

It is helpful to be conversant in both languages.

$$\phi \stackrel{\{Y_{\ell m}\}}{\Longrightarrow} r + (\theta, \varphi)$$

$$\phi \stackrel{STF}{\Longrightarrow} (x, y, z)$$

Taylor expansion of the external potential

The integral solution of (5)[Appendix III]:

$$\phi(t,\vec{r}) = -G\int_v \frac{\rho(t,\vec{r}')}{|\vec{r}-\vec{r}'|}d^3\vec{r}' \tag{32}$$

Consider a field point $\vec{r}$ that lies outside the matter distribution. With $|\vec{r}'| < |\vec{r}|$, we carry out a Taylor expansion of $|\vec{r}-\vec{r}'|^{-1}$ in powers of $\vec{r}'$:

$$\begin{aligned}\frac{1}{|\vec{r}-\vec{r}'|} &= \frac{1}{r} - x'^j\partial_j\left(\frac{1}{r}\right) + \frac{1}{2}x'^jx'^k\partial_j\partial_k\left(\frac{1}{r}\right) - \cdots \\ &= \frac{1}{r} - x'^j\partial_j\left(\frac{1}{r}\right) + \frac{1}{2}x'^{jk}\partial_{jk}\left(\frac{1}{r}\right) - \cdots \\ &= \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}x'^L\partial_L\left(\frac{1}{r}\right)\end{aligned} \tag{33}$$

with $x^L = x^{j_1j_2\cdots j_\ell} = x^{j_1}x^{j_2}\cdots x^{j_\ell}$, $\partial_L = \partial_{j_1j_2\cdots j_\ell} = \partial_{j_1}\partial_{j_2}\cdots\partial_{j_\ell}$

Taylor expansion of the external potential

Substituting (33) into (32) gives

$$\begin{aligned}\phi_{ext}(t,\vec{r}) &= -G\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{\ell!}I^{L}\partial_{L}\left(\frac{1}{r}\right) \\ &\stackrel{?}{=} -G\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{\ell!}I^{\langle L\rangle}\partial_{\langle L\rangle}\left(\frac{1}{r}\right)\end{aligned} \tag{34}$$

with

$$I^{\langle L\rangle}(t) := \int \rho(t,\vec{r}')x'^{\langle L\rangle}d^{3}\vec{r}' \tag{35}$$

$$Y_{\ell m}/r^{\ell+1} \longleftrightarrow \partial_{\langle L\rangle}r^{-1}$$

$$I_{\ell m} \longleftrightarrow I^{\langle L\rangle}$$

STF combinations

$\vec{n} := \vec{r}/r$,

$$\partial_j r = n_j$$
$$\partial_j n_k = \partial_k n_j = \frac{1}{r}\left(\delta_{jk} - n_j n_k\right) \tag{36}$$

We compute the derivatives of r^{-1} by making repeated use of (36)

$$\partial_j r^{-1} = -n_j r^{-2}$$
$$\partial_{jk} r^{-1} = \left(3 n_j n_k - \delta_{jk}\right) r^{-3}$$

$$\partial_{jkn} r^{-1} = -\left[15 n_j n_k n_n - 3\left(n_j \delta_{kn} + n_k \delta_{jn} + n_n \delta_{jk}\right)\right] r^{-4} \tag{37}$$

Obviously they are symmetric and tracefree except $\partial_j r^{-1}$.
$(eg.\ \delta^{jk}\partial_{jkn} r^{-1} = \nabla^2 \partial_n r^{-1} = \partial_n \nabla^2 r^{-1} = 0)$
We conclude that $\partial_L r^{-1}$ is an STF tensor. $\left(\partial_L r^{-1} = \partial_{\langle L \rangle} r^{-1}\right)$

STF combinations

Conventionally, STF products of vectors such as n^j are obtained by **beginning with the "raw" products** $n^j n^k \cdots$ and **removing all traces, maintaining symmetry on all indices**. Explicit examples are

$$
\begin{aligned}
n^{\langle jk\rangle} =& n^j n^k - \frac{1}{3}\delta^{jk} \\
n^{\langle jkn\rangle} =& n^j n^k n^n - \frac{1}{5}\left(\delta^{jk} n^n + \delta^{jn} n^k + \delta^{kn} n^j\right) \\
n^{\langle jknp\rangle} =& n^j n^k n^n n^p - \frac{1}{7}(\delta^{jk} n^n n^p + \delta^{jn} n^k n^p + \delta^{jp} n^k n^n + \delta^{kn} n^j n^p \\
&+ \delta^{kp} n^j n^n + \delta^{np} n^j n^k) + \frac{1}{35}(\delta^{jk}\delta^{np} + \delta^{jn}\delta^{kp} + \delta^{jp}\delta^{kn})
\end{aligned}
\tag{38}
$$

STF combinations

General formula for such STF products:

$$n^{\langle L\rangle} = n^{\langle j_1 j_2 \cdots j_\ell\rangle} = \sum_{p=0}^{[\ell/2]} (-1)^p \frac{\ell!(2\ell-2p-1)!!}{(\ell-2p)!(2\ell-1)!!(2p)!!} \times \delta^{(j_1 j_2}\delta^{j_3 j_4}\ldots\delta^{j_{2p-1}j_{2p}} n^{j_{2p+1}} n^{j_{2p+2}}\ldots n^{j_\ell)} \tag{39}$$

The number of the independent components of $n^{\langle L\rangle}$:

$$3^\ell \xrightarrow{symm} \frac{(\ell+1)(\ell+2)}{2} \xrightarrow{tracefree} 2\ell+1$$

· STF identities:

$$n'_{\langle L\rangle} n^{\langle L\rangle} = \frac{\ell!}{(2\ell-1)!!} P_\ell(\mu) \tag{40}$$

where $\mu := \vec{n}\cdot\vec{n}'$.

STF combinations

Comparing (37) and (38), we find that $\partial_j r^{-1} = -n_j r^{-2}$, $\partial_{jk} r^{-1} = 3n_{\langle jk \rangle} r^{-3}$,and $\partial_{jkn} r^{-1} = -15 n_{\langle jkn \rangle} r^{-4}$.

The general rule can be obtained by induction:

$$\partial_L r^{-1} = \partial_{\langle L \rangle} r^{-1} = (-1)^\ell (2\ell - 1)!! \frac{n_{\langle L \rangle}}{r^{\ell+1}} \tag{41}$$

Now we can express (34) as

$$\phi_{ext}(t, \vec{r}) = -G \sum_{\ell=0}^{\infty} \frac{(2\ell - 1)!!}{\ell!} I^{\langle L \rangle} \frac{n_{\langle L \rangle}}{r^{\ell+1}} \tag{42}$$

and explain the reason for the angular brackets on $I^{\langle L \rangle}$.

STF combinations

$I^L := \int \rho' x'^L d^3\vec{r}'$ denotes the "raw" multipole moments. In view of (39), I^L differs from $I^{\langle L\rangle}$ by **a sum of terms involving Kronecker deltas**, and these automatically give zero when multiplied by the tracefree $\partial_L r^{-1}$.$\left(Hint: \ \cdots \delta^{j_m j_n}\partial_L r^{-1} = \cdots \partial_{L-2}\nabla^2 r^{-1} = 0\right)$
As a result, we find that $I^L \partial_L r^{-1} = I^{\langle L\rangle}\partial_{\langle L\rangle} r^{-1}$

Generally, whenever an arbitrary tensor A^L multiplies an STF tensor $B_{\langle L\rangle}$, the outcome is

$$A^L B_{\langle L\rangle} = A^{\langle L\rangle} B_{\langle L\rangle} \tag{43}$$

where $A^{\langle L\rangle}$ is the tensor obtained from A^L by complete symmetrization and removal of all traces.

From STF to $\{ Y_{\ell m} \}$

Using STF identities (40), we rewrite (42):

$$\begin{aligned} \phi &= -G \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \int \rho' x'^{\langle L \rangle} d^3\vec{r}' \frac{n_{\langle L \rangle}}{r^{\ell+1}} \\ &= -G \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \int \rho' r'^{\ell} n'^{\langle L \rangle} n_{\langle L \rangle} d^3\vec{r}' r^{-(\ell+1)} \\ &= -G \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \int \rho' r'^{\ell} \frac{\ell!}{(2\ell-1)!!} P_\ell(\mu) d^3\vec{r}' r^{-(\ell+1)} \end{aligned} \tag{44}$$

From STF to $\{Y_{\ell m}\}$

Since $P_\ell(\mu) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi)$,

$$\begin{aligned}
\phi &= -G \sum_{\ell m} \frac{4\pi}{2\ell+1} \int_0^R r'^{\ell} \left(\int \rho' Y^*_{\ell m}(\theta', \varphi')\, d\Omega' \right) r'^2 dr' \frac{Y_{\ell m}(\theta, \varphi)}{r^{\ell+1}} \\
&= -G \sum_{\ell m} \frac{4\pi}{2\ell+1} \int_0^R r'^{\ell} \rho'_{\ell m} r'^2 dr' \frac{Y_{\ell m}(\theta, \varphi)}{r^{\ell+1}} \\
&= -G \sum_{\ell m} \frac{4\pi}{2\ell+1} q_{\ell m}(t, R) \frac{Y_{\ell m}(\theta, \varphi)}{r^{\ell+1}}
\end{aligned} \tag{45}$$

Appendix I

$$\begin{aligned}\vec{\nabla} \times \vec{E} &= \vec{\nabla} \times \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}')}{\eta^2} \hat{\eta} d^3\vec{r}' \\ &= \frac{1}{4\pi\epsilon_0} \int_V \left[\vec{\nabla} \times \left(\frac{\hat{\eta}}{\eta^2} \right) \right] \rho(\vec{r}') d^3\vec{r}' \end{aligned} \tag{46}$$

Using Cartesian coordinate bases and calculating the components:

$$\begin{aligned}\left[\vec{\nabla} \times \left(\frac{\hat{\eta}}{\eta^2} \right) \right]^i &= \epsilon^{ij}{}_k \frac{\partial}{\partial x^j} \frac{x^k - x'^k}{\eta^3} \\ &= \epsilon^{ij}{}_k \left(\frac{\delta^k{}_j}{\eta^3} + \left(x^k - x'^k \right) \frac{-3}{\eta^4} \frac{\partial \eta}{\partial x^j} \right) \\ &= \frac{\epsilon^{ij}{}_j}{\eta^3} + \epsilon^{ij}{}_k \left(x^k - x'^k \right) \frac{-3}{\eta^4} \frac{x_j - x'_j}{\eta} \end{aligned} \tag{47}$$

The first term of (47) is obviously zero.

Appendix I

$$\begin{aligned}
(\textit{The second term}) &= \frac{-3}{\eta^5}\epsilon^{ij}{}_k(x^k - x'^k)(x_j - x'_j) \\
&= \frac{-3}{\eta^5}\epsilon^{ij}{}_k(x^k - x'^k)(x^l - x'^l)\delta_{lj} \\
&= \frac{-3}{\eta^5}\epsilon^{i}{}_{lk}(x^k - x'^k)(x^l - x'^l) \\
&\overset{l\leftrightarrow k}{=} \frac{-3}{\eta^5}\epsilon^{i}{}_{kl}(x^l - x'^l)(x^k - x'^k) \\
&= \frac{-3}{\eta^5}\left(-\epsilon^{i}{}_{lk}\right)(x^k - x'^k)(x^l - x'^l) \\
&= -(\textit{The second term})
\end{aligned} \tag{48}$$

So the second term of (47) is also zero.

$$\rightarrow \left[\vec{\nabla}\times\left(\frac{\hat{\eta}}{\eta^2}\right)\right]^i = 0 \rightarrow \vec{\nabla}\times\vec{E} = 0$$

Appendix II

We take the divergence of the both sides of (3)

$$\begin{aligned}-\vec{\nabla}^2\varphi = \vec{\nabla}\cdot\vec{E} &= \vec{\nabla}\cdot\left(\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}')}{\eta^2}\hat{\eta}d^3\vec{r}'\right) \\ &= \frac{1}{4\pi\epsilon_0}\int_V \left[\vec{\nabla}\cdot\left(\frac{\hat{\eta}}{\eta^2}\right)\right]\rho(\vec{r}')d^3\vec{r}'\end{aligned} \tag{49}$$

If $\eta \neq 0$,

$$\begin{aligned}\vec{\nabla}\cdot\left(\frac{\hat{\eta}}{\eta^2}\right) &= \frac{\partial}{\partial x^i}\left(\frac{x^i - x'^i}{\eta^3}\right) \\ &= \frac{\delta^i{}_i}{\eta^3} + (x^i - x'^i)\,\frac{-3}{\eta^4}\,\frac{x_i - x'_i}{\eta} \\ &= \frac{3}{\eta^3} - \frac{3\eta^2}{\eta^5} = 0\end{aligned} \tag{50}$$

So $\vec{\nabla}\cdot\left(\frac{\hat{\eta}}{\eta^2}\right)$ is zero everywhere except $\vec{r} = \vec{r}'$.

Appendix II

Now we calculate a volume integral in a sphere of radius R,centered at $\vec{r}'$

$$\begin{aligned}\int_V \vec{\nabla}\cdot\left(\frac{\hat{\eta}}{\eta^2}\right) d^3\vec{r} &= \oint_{\partial V} \frac{\hat{\eta}}{\eta^2}\cdot d\vec{S} \\ &= \int_0^{2\pi} d\varphi \int_0^{\pi} \frac{1}{R^2} R^2 \sin\theta d\theta \\ &= 4\pi\end{aligned} \tag{51}$$

Therefore,

$$\vec{\nabla}\cdot\left(\frac{\hat{\eta}}{\eta^2}\right) = 4\pi\delta^3\left(\vec{\eta}\right) \tag{52}$$

Appendix II

The right-hand side of (49):

$$\frac{1}{4\pi\epsilon_0}\int_V \left[\vec{\nabla}\cdot\left(\frac{\hat{\eta}}{\eta^2}\right)\right]\rho(\vec{r}')d^3\vec{r}' = \frac{1}{4\pi\epsilon_0}\int_V \left[4\pi\delta^3\left(\vec{\eta}\right)\right]\rho(\vec{r}')d^3\vec{r}' = \frac{\rho\left(\vec{r}\right)}{\epsilon_0} \tag{53}$$

So (49) becomes:

$$\vec{\nabla}^2\varphi = -\frac{\rho\left(\vec{r}\right)}{\epsilon_0} \tag{54}$$

Appendix III

Method 1: Green's Function[1]
Method 2: Fourier Transform

$$\vec{\nabla}^2\phi = 4\pi G\rho\left(t, \vec{r}\right) \tag{55}$$

Using Fourier's trick,
$\mathscr{F}\left[\vec{\nabla}^2\phi\left(t, \vec{r}\right)\right] = \left[(ik_1)^2 + (ik_2)^2 + (ik_3)^2\right]\Phi\left(t, \vec{k}\right)$
$\mathscr{F}\left[\rho\left(t, \vec{r}\right)\right] = R\left(t, \vec{k}\right)$
We get

$$\Phi\left(t, \vec{k}\right) = -\frac{4\pi G}{k^2}R\left(t, \vec{k}\right) \tag{56}$$

Appendix III

Now performing an inverse Fourier transform

$$\begin{aligned}
\phi(t,\vec{r}) &= \mathscr{F}^{-1}\left[\Phi\left(t,\vec{k}\right)\right] \\
&= \frac{-4\pi G}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\frac{1}{k^2}R\left(t,\vec{k}\right)e^{i\vec{k}\cdot\vec{r}}d^3\vec{k} \\
&= \frac{-4\pi G}{(2\pi)^{3}}\int_{-\infty}^{\infty}\frac{1}{k^2}\int_{-\infty}^{\infty}\rho(t,\vec{r}\,')e^{-i\vec{k}\cdot\vec{r}\,'}d^3\vec{r}\,'e^{i\vec{k}\cdot\vec{r}}d^3\vec{k} \\
&= \frac{-4\pi G}{(2\pi)^{3}}\int_{-\infty}^{\infty}\rho(t,\vec{r}\,')d^3\vec{r}\,'\int_{-\infty}^{\infty}\frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{k^2}d^3\vec{k}
\end{aligned} \tag{57}$$

The internal integral can be calculated by using spherical coordinates.

Appendix III

Making the direction of k_3 parallel to $\vec{r}-\vec{r}'$,then

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{k^2} d^3\vec{k} &= \int_0^{+\infty} dk \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta e^{ik|\vec{r}-\vec{r}'|\cos\theta} \\
&= -2\pi \int_0^{+\infty} \left. \frac{e^{ik|\vec{r}-\vec{r}'|\cos\theta}}{ik|\vec{r}-\vec{r}'|} \right|_{\theta=0}^{\theta=\pi} dk \\
&= 4\pi \int_0^{+\infty} \frac{\sin\left(k|\vec{r}-\vec{r}'|\right)}{k|\vec{r}-\vec{r}'|} dk \\
&= \frac{4\pi}{|\vec{r}-\vec{r}'|} \int_0^{+\infty} \frac{\sin\alpha}{\alpha} d\alpha \\
&= \frac{2\pi^2}{|\vec{r}-\vec{r}'|}
\end{aligned}
\tag{58}
$$

Appendix III

Combining (57) and (58), we get

$$\phi(t,\vec{r}') = -G\int_V \frac{\rho(t,\vec{r}')}{|\vec{r}-\vec{r}'|}d^3\vec{r}' \tag{59}$$

Appendix IV

Spherical harmonics satisfy the eigenvalue equation:

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y_{\ell m}=-\ell(\ell+1)Y_{\ell m} \tag{60}$$

they are given explicitly by

$$Y_{\ell m}(\theta,\varphi)=\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\varphi} \tag{61}$$

where

$$P_\ell^m(x):=(-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_\ell(x) \tag{62}$$

$$P_\ell(x):=\frac{1}{2^\ell\ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell \tag{63}$$

Appendix IV

Orthonormalization relation:

$$\int Y_{\ell m} Y^*_{\ell' m'} d\Omega = \delta_{\ell\ell'}\delta_{mm'} \tag{64}$$

Closure relation:

$$\begin{aligned}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)Y^*_{\ell m}(\theta',\varphi') &= \delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi') \\ &= \frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\varphi-\varphi')\end{aligned} \tag{65}$$

Spherical-harmonic decomposition:

$$\begin{aligned}f(\theta,\varphi) &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell m}(\theta,\varphi), \\ f_{\ell m} &= \int f(\theta,\varphi)Y^*_{\ell m}(\theta,\varphi)d\Omega\end{aligned} \tag{66}$$

Reference

[1] Qiao Gu.
Mathematical methods for physics.
pages 366–374, 2012.

Thanks!