Assignment 1

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Exercise 1

As you might have noticed, we freely use numerous basic notions from mechanics, electrodynamics, quantum mechanics, and special relativity.
If you feel a little unsure about some of this, take some time to review the necessary material from your old courses (or elsewhere — the web is often a valuable source of information).
Do not hesitate to discuss and ask questions to your colleagues.

Exercise 2

Carefully study chapter 2 in **Mandl & Shaw**.
Pay particular attention to section 2.4 where the relation between constants of motion, divergence-less currents and tensors, and symmetries are studied.
This chapter is very essential, so please devote sufficient time to it.

As an application, show that for the (gauge-fixed) Maxwell theory with Lagrange density (function)

$$\mathcal{L} = -\tfrac{1}{2}\, \partial_\mu A_\nu\, \partial^\mu A^\nu$$

the energy-momentum tensor is

$$T^{\mu\nu} = -\partial^\mu A^\rho\, \partial^\nu A_\rho + \tfrac{1}{2}\eta^{\mu\nu}\, \partial_\rho A_\sigma\, \partial^\rho A^\sigma$$

Using the equations of motion following from the Lagrangian above (calculate them!), show that

$$\partial_\nu T^{\mu\nu} = 0$$

indeed holds.
What are, for this theory, the expressions for the **energy E** and the **momentum** $\vec{p}$?

Solution 2

I. Energy Momentum Tensor from Lagrangian Density

(0) canonical Noether energy–momentum tensor

We use the **canonical Noether energy–momentum tensor** (from spacetime translations):

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)}\,\partial^\nu A_\rho - \eta^{\mu\nu}\,\mathcal{L}$$

(1) the derivative term

From the given Lagrangian,

$$\mathcal{L} = -\tfrac{1}{2}\,\partial_\alpha A_\beta\,\partial^\alpha A^\beta$$

Plug into the first term of the definition of energy-momentum tensor

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} = -\,\frac{1}{2}\left[\frac{\partial(\partial_\alpha A_\beta\,\partial^\alpha A^\beta)}{\partial(\partial_\mu A_\rho)}\right] = -\,\partial^\mu A^\rho$$

(2) result

Hence

$$T^{\mu\nu} = \big(-\,\partial^\mu A^\rho\big)\,\partial^\nu A_\rho - \eta^{\mu\nu}\mathcal{L}.$$

and finally as we insert Lagrangian density function explicitly, the energy-momentum tensor is explicitly:

$$T^{\mu\nu} = -\,\partial^\mu A^\rho\,\partial^\nu A_\rho + \frac{1}{2}\eta^{\mu\nu}\,\partial_\alpha A_\beta\,\partial^\alpha A^\beta$$

II. Energy-momentum Conservation on the Equation of Motion

(1) Euler–Lagrange equations from Lagrangian Density claims $\Box A^\nu = 0$

Treat A_ν as the set of fields. Since $\mathcal{L}$ has no explicit A_ν dependence,

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = 0, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\,\partial^\mu A^\nu$$

Inserting these into Euler–Lagrange equation then claims:

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \quad \Rightarrow \quad \partial_\mu(-\partial^\mu A^\nu) = 0 \quad \Rightarrow \quad \Box A^\nu \equiv \partial_\mu \partial^\mu A^\nu = 0$$

So each component satisfies the wave equation (this corresponds to the Lorenz gauge–fixed, free Maxwell theory).

(2) Divergencelessness of energy-momentum tensor

From I. Energy Momentum Tensor from Lagrangian Density we have the explicit form of energy momentum tensor in terms of field:

$$T^{\mu\nu} = -\,\partial^\mu A^\rho\,\partial^\nu A_\rho + \frac{1}{2}\,\eta^{\mu\nu}\,\partial_\rho A_\sigma\,\partial^\rho A^\sigma. \tag{$\bigstar$}$$

and we want to **prove:** $\partial_\nu T^{\mu\nu} = 0$ on the equations of motion.

$$\begin{aligned}\partial_\nu T^{\mu\nu} &= -\,\partial_\nu\big(\partial^\mu A^\rho\,\partial^\nu A_\rho\big) + \frac{1}{2}\,\partial^\mu\big(\partial_\rho A_\sigma\,\partial^\rho A^\sigma\big) \\ &= -\,(\partial_\nu \partial^\mu A^\rho)\,\partial^\nu A_\rho - \,\partial^\mu A^\rho\,\underbrace{\partial_\nu \partial^\nu A_\rho}_{\Box A_\rho} + \frac{1}{2}\,\partial^\mu\big(\partial_\rho A_\sigma\,\partial^\rho A^\sigma\big).\end{aligned}$$

where the first equality is just definitions. And as we get the second equality, we **notice** that on the EOM, $\Box A_\rho = 0$ (cuz we chose the Lorentz gauge?), so the middle term vanishes; on the last term use Leibniz and commuting derivatives in flat space:

$$\frac{1}{2}\,\partial^\mu(\partial_\rho A_\sigma\,\partial^\rho A^\sigma) = \frac{1}{2}\big[(\partial^\mu \partial_\rho A_\sigma)(\partial^\rho A^\sigma) + (\partial_\rho A_\sigma)(\partial^\mu \partial^\rho A^\sigma)\big] = (\partial_\rho \partial^\mu A_\sigma)(\partial^\rho A^\sigma),$$

where in the last step we relabeled dummy indices and used symmetry.
Rename indices $\rho \to \nu$, $\sigma \to \rho$ to match the first term:

$$\frac{1}{2}\,\partial^\mu(\cdots) = (\partial_\nu \partial^\mu A^\rho)\,\partial^\nu A_\rho.$$

This cancels the first term remaining and thus:

$$\partial_\nu T^{\mu\nu} = -(\partial_\nu \partial^\mu A^\rho)\,\partial^\nu A_\rho + (\partial_\nu \partial^\mu A^\rho)\,\partial^\nu A_\rho = 0.$$

Expression for Energy and momentum

(0) Definition of the conserved 4-momentum

Define the conserved 4-momentum as

$$P^\nu = \int d^3x\, T^{0\nu}$$

(1) Energy as the time component of 4-momentum

$$E \equiv P^0 = \int d^3x\, T^{00}, \qquad T^{00} = -\,\partial^0 A^\rho\,\partial^0 A_\rho + \frac{1}{2}\,\eta^{00}\,\partial_\rho A_\sigma\,\partial^\rho A^\sigma$$

(2) Momentum as the spacial part of 4-momentum

Since $\eta^{0i} = 0$,

$$T^{0i} = -\,\partial^0 A^\rho\,\partial^i A_\rho, \qquad p^i \equiv P^i = \int d^3x\, T^{0i}$$

> Please find explicit form in hand-written version

Exercise 3

Given a set of fields ϕ_a, $a \in \{1, \cdots, n\}$, with a Lagrange density that depends only on the fields themselves and their first-order derivatives, i.e.,

$$\mathcal{L} = \mathcal{L}(\phi, \partial\phi).$$

- **Show** that by varying the action $S = c^{-1}\int d^4x\,\mathcal{L}$, the equations of motion (Euler–Lagrange equations) are

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0.$$

- **Now rederive** these equations of motion by discretizing space (approximate space by a lattice with lattice spacing ℓ and at the end take the limit $\ell \to 0$) and using the standard expression for the Euler–Lagrange equations.
  Some more details: discretize "space" by a lattice with side length ℓ.
  The fields on the lattice are given by $\phi_a^{(i,j,k)}(t)$, $\{i,j,k\} \in \mathbb{Z}^3$, such that in the limit $\ell \to 0$ we have

$$\lim_{\ell\to 0}\phi_a^{(i,j,k)}(t) = \phi_a(t,\vec{x}).$$

  Upon discretization of space the Lagrangian becomes (see the lecture)

$$L = \sum_{(i,j,k)\in\mathbb{Z}^3} \ell^3\,\mathcal{L}^{(i,j,k)},$$

  which in the limit $\ell \to 0$ gives

$$L = \int d^3\vec{x}\,\mathcal{L}.$$

  From mechanics, the Euler–Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}_a^{(i,j,k)}} - \frac{\partial L}{\partial\phi_a^{(i,j,k)}} = 0.$$

  Analyze this equation carefully and show that it indeed reduces to

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0, \tag{10}$$

  in the limit $\ell \to 0$.

Solution 3

I. Field EOM from Variation of Action

(0) Setup

Start from

$$S[\phi] = c^{-1} \int d^4x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a)$$

We perform an infinitesimal variation $\phi_a \to \phi_a + \delta\phi_a$:

$$\delta S = c^{-1} \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \, \delta(\partial_\mu \phi_a) \right)$$

(1) Second term integrate by parts

Integrate the second term by parts:

$$\int d^4x \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \, \delta(\partial_\mu \phi_a) = \int d^4x \, \left[-\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \delta\phi_a \right] + \text{surface term}$$

we can drop the surface term (since $\delta\phi_a$ vanishes on the boundary)

(2) Combine and use extremal condition

Combining two terms yields the total variation of action:

$$\delta S = c^{-1} \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right] \delta\phi_a$$

Then by setting $\delta S = 0$ for arbitrary $\delta\phi_a$ gives the Euler–Lagrange equations:

$$\boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0.}$$

II. Field EOM from the Continuum Limit of Classical EL equation

(0) Motivation

- In classical mechanics, the Euler–Lagrange (EL) equation follows from minimizing the **action**

$$S = \int L(q_i, \dot{q}_i, t)\, dt,$$

yielding

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

- In field theory, each **field value at a point in space** plays the role of a coordinate q_i. Thus, the field equation of motion may arise as the **continuum limit** of the EL equations for a system with infinitely many coordinates.

(1) Setup

- Cubic lattice of spacing ℓ, sites $\vec{i} = (i, j, k) \in \mathbb{Z}^3$, positions $\vec{x}_{\vec{i}} = \ell\, \vec{i}$.
- Fields on sites: $\phi_a^{\vec{i}}(t) \approx \phi_a(t, \vec{x}_{\vec{i}})$.
- Use the **central difference** for spatial derivatives (consistent and $\mathcal{O}(\ell^2)$):

$$(\partial_m \phi_a)^{\vec{i}} \;\approx\; \frac{\phi_a^{\vec{i}+\hat{m}} - \phi_a^{\vec{i}-\hat{m}}}{2\ell}, \qquad m \in \{x, y, z\}$$

- Discrete Lagrangian:

$$L[\phi, \dot{\phi}] \;=\; \sum_{\vec{i}} \ell^3\, \mathcal{L}^{\vec{i}} \quad \text{with} \quad \mathcal{L}^{\vec{i}} = \mathcal{L}\big(\phi_a^{\vec{i}},\; \dot{\phi}_a^{\vec{i}},\; (\partial_m \phi_a)^{\vec{i}}\big)$$

From standard mechanics, for any coordinate $\phi_a^{(i,j,k)}(t)$,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_a^{(i,j,k)}}\right) - \frac{\partial L}{\partial \phi_a^{(i,j,k)}} = 0$$

Substitute $L = \sum_{(i,j,k)} \ell^3\, \mathcal{L}^{(i,j,k)}$:

$$\frac{d}{dt}\left(\ell^3\, \frac{\partial \mathcal{L}^{(i,j,k)}}{\partial \dot{\phi}_a^{(i,j,k)}}\right) - \ell^3\, \frac{\partial \mathcal{L}^{(i,j,k)}}{\partial \phi_a^{(i,j,k)}} = 0$$

Dividing by ℓ^3,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}^{(i,j,k)}}{\partial \dot{\phi}_a^{(i,j,k)}}\right) - \frac{\partial \mathcal{L}^{(i,j,k)}}{\partial \phi_a^{(i,j,k)}} = 0$$

(2) Recovering spatial derivatives in the continuum limit

In the discretized picture, the **spatial derivatives** inside $\mathcal{L}$ are replaced by finite differences.
For example,

$$\partial_x \phi_a(t, \vec{x}_{i,j,k}) \;\approx\; \frac{\phi_a^{(i+1,j,k)}(t) - \phi_a^{(i-1,j,k)}(t)}{2\ell}$$

The Lagrangian density therefore depends on the neighboring lattice sites:

$$\mathcal{L}^{(i,j,k)} = \mathcal{L}\left(\phi_a^{(i,j,k)},\ \frac{\phi_a^{(i+1,j,k)} - \phi_a^{(i-1,j,k)}}{2\ell},\ \frac{\phi_a^{(i,j+1,k)} - \phi_a^{(i,j-1,k)}}{2\ell},\ \frac{\phi_a^{(i,j,k+1)} - \phi_a^{(i,j,k-1)}}{2\ell}\right)$$

When we differentiate $\mathcal{L}^{(i,j,k)}$ with respect to $\phi_a^{(i,j,k)}$, it picks up contributions not only from the point itself but also from the neighboring points, because $\phi_a^{(i,j,k)}$ appears in the finite-difference expressions.

Collecting contributions from all spatial directions and time:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}^{(i,j,k)}}{\partial \dot{\phi}_a}\right) \;\longrightarrow\; \partial_0\left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)}\right)$$

and

$$\frac{1}{\ell}\left[\frac{\partial \mathcal{L}^{(i+1,j,k)}}{\partial(\partial_x \phi_a)} - \frac{\partial \mathcal{L}^{(i-1,j,k)}}{\partial(\partial_x \phi_a)}\right] \;\longrightarrow\; \partial_x\left(\frac{\partial \mathcal{L}}{\partial(\partial_x \phi_a)}\right)$$

and similarly for y, z.

Thus the discrete Euler–Lagrange equations become, in the limit $\ell \to 0$,

$$\boxed{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0}$$

which is exactly the **field-theoretic Euler–Lagrange equation**.

Exercise 4

Given a scalar field $\phi(\vec{x}, t)$ with Lagrange density

$$\mathcal{L} = \tfrac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}\,\frac{m^2c^2}{\hbar^2}\,\phi^2 - V(\phi),$$

where $V(\phi)$ is a real function of ϕ (often called the **potential**):

1. Calculate the equations of motion by varying the action.
2. Do you recognize this equation when $V(\phi) = 0$?

Solution 4

I. EOM (EL Equations) from Action

(1) Derive EL Equation by variantion

The action is defined by

$$S = \frac{1}{c}\int d^4x\,\mathcal{L}(\phi, \partial_\mu\phi)$$

Vary with respect to ϕ:

$$\begin{aligned}\delta S &= \frac{1}{c}\int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta(\partial_\mu\phi)\right]\\ &= \frac{1}{c}\int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu(\delta\phi)\right]\end{aligned}$$

Integrate by parts the second term (and drop the boundary term, since $\delta\phi = 0$ at infinity):

$$\delta S = \frac{1}{c}\int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\delta\phi$$

Setting $\delta S = 0$ for arbitrary $\delta\phi$ yields the Euler–Lagrange equation:

$$\boxed{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0}$$

(2) Insert Lagrangian density into EL equation and get explicit EOM

1. **Derivative with respect to $\partial_\mu \phi$:**

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

2. **Derivative with respect to ϕ:**

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{m^2 c^2}{\hbar^2}\phi - V'(\phi)$$

   where $V'(\phi) \equiv dV/d\phi$.

Plug into the Euler–Lagrange equation:

$$\partial_\mu \partial^\mu \phi + \frac{m^2 c^2}{\hbar^2}\phi + V'(\phi) = 0$$

(3) Packaging

The operator $\partial_\mu \partial^\mu$ is the **d'Alembertian** in Minkovskian space:

$$\Box \equiv \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2.$$

Therefore, the field equation can be written as

$$\boxed{\Box \phi + \frac{m^2 c^2}{\hbar^2}\phi + V'(\phi) = 0.}$$

II. Special case: $V(\phi) = 0$: KG Equation for a free massive scalar field

If $V(\phi) = 0$, then $V'(\phi) = 0$, and we obtain

$$\Box \phi + \frac{m^2 c^2}{\hbar^2}\phi = 0$$

This is the **Klein–Gordon equation** for a free massive scalar field:

$$\boxed{(\Box + m^2 c^2/\hbar^2)\phi = 0.}$$

Exercise 5

Extend the one-dimensional harmonic oscillator analysis from class to the n-dimensional isotropic harmonic oscillator with Hamiltonian

$$\hat{H} = \sum_{j=1}^{n} \left(\frac{1}{2m} \hat{p}_j^{\,2} + \frac{k}{2} \hat{q}_j^{\,2} \right),$$

nd commutation relations

$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{q}_i, \hat{p}_j] = i\hbar\, \delta_{ij}.$$

1. Determine operators a_i and $a_i^\dagger$ as linear combinatio ns of $\hat{q}_i$ and $\hat{p}_i$ such that
   $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0,$
   $[a_i, a_j^\dagger] = \delta_{ij}.$
2. Rewrite $\hat{H}$ in terms of these operators. Determine the ground state and energy spectrum. Note the degeneracies at each energy level.
3. Show that the n^2 operators $a_i^\dagger a_j$ commute with $\hat{H}$.
   Determine the **Cartan subalgebra**, i.e. the maximal abelian subset of these operators (there are n of them).
   Show that the energy eigenstates are also eigenstates of this abelian subset.

Solution 5

I. Determine Ladder Operators

(1) Recall 1-dimenssional solution

In one dimension,

$$a = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} + \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}, \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} - \frac{i}{\sqrt{2m\hbar\omega}}\hat{p},$$

where $\omega = \sqrt{k/m}$.

(2) Generalize to n-dimenssion and verify comutation relation

We generalize this to each coordinate $i = 1, \ldots, n$:

$$a_i = \sqrt{\frac{m\omega}{2\hbar}}\hat{q}_i + \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}_i, \qquad a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{q}_i - \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}_i.$$

To verify these definitions satisfy the commutation relation, we compute

$$[a_i, a_j] = \frac{m\omega}{2\hbar}[\hat{q}_i, \hat{q}_j] + \frac{i^2}{2m\hbar\omega}[\hat{p}_i, \hat{p}_j] + \frac{i}{2\hbar}([\hat{q}_i, \hat{p}_j] + [\hat{p}_i, \hat{q}_j]).$$

of which the first two terms are trivial, and since $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$, the last term cancels:
and thus $[a_i, a_j = 0]$;
Similarly, $[a_i^\dagger, a_j^\dagger] = 0$;
Then we check the last relation where

$$[a_i, a_j^\dagger] = \left[\sqrt{\tfrac{m\omega}{2\hbar}}\hat{q}_i + \tfrac{i}{\sqrt{2m\hbar\omega}}\hat{p}_i,\ \sqrt{\tfrac{m\omega}{2\hbar}}\hat{q}_j - \tfrac{i}{\sqrt{2m\hbar\omega}}\hat{p}_j\right]$$

and thus

$$[a_i, a_j^\dagger] = \frac{m\omega}{2\hbar}[\hat{q}_i, \hat{q}_j] + \frac{i^2}{2m\hbar\omega}[\hat{p}_i, \hat{p}_j] + \frac{i}{2\hbar}([\hat{q}_i, \hat{p}_j] - [\hat{p}_i, \hat{q}_j]) = \delta_{ij}$$

.

II. Express $\hat{H}$ in terms of a_i and solve for Energy Spectrum

(0) Hamiltonian in terms of canonical observables

Hamiltonian of QHO is defined by:

$$\hat{H} = \sum_{j=1}^{n}\left(\frac{1}{2m}\hat{p}_j^{\,2} + \frac{k}{2}\hat{q}_j^{\,2}\right), \qquad [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{q}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$$

(1) Express $\hat{p}_i, \hat{q}_i$ in terms of a_i

From the definitions we invert:

$$\hat{q}_i = \sqrt{\frac{\hbar}{2m\omega}}(a_i + a_i^\dagger), \qquad \hat{p}_i = i\sqrt{\frac{\hbar m\omega}{2}}(a_i^\dagger - a_i)$$

(2) Rewrite $\hat{H}$

Substitute (1) into (0) term by term:

$$\frac{1}{2m}\hat{p}_i^2 = \frac{\hbar\omega}{4}(a_i^\dagger - a_i)^2, \qquad \frac{k}{2}\hat{q}_i^2 = \frac{m\omega^2}{2}\frac{\hbar}{2m\omega}(a_i + a_i^\dagger)^2 = \frac{\hbar\omega}{4}(a_i + a_i^\dagger)^2.$$

Add together:

$$\frac{1}{2m}\hat{p}_i^2 + \frac{k}{2}\hat{q}_i^2 = \frac{\hbar\omega}{4}\left[(a_i^\dagger - a_i)^2 + (a_i + a_i^\dagger)^2\right] = \frac{\hbar\omega}{2}(a_i a_i^\dagger + a_i^\dagger a_i)$$

Using commutation relation $a_i a_i^\dagger = a_i^\dagger a_i + 1$,

$$\hat{H} = \hbar\omega \sum_{i=1}^{n}\left(a_i^\dagger a_i + \tfrac{1}{2}\right) =: \hbar\omega\left(N + \tfrac{n}{2}\right)$$

where the second equality is the definition of **total number operator**:

$$N = \sum_{i=1}^{n} a_i^\dagger a_i$$

(3) Ground state energy

The **ground state** $|0\rangle$ is defined as the state annihilated by all a_i:

$$a_i|0\rangle = 0, \qquad i = 1, \ldots, n$$

We then check this is indeed an energy eigenstate: since by this defintion we have

$$N|0\rangle = \left(\sum_i a_i^\dagger a_i\right)|0\rangle = 0$$

hence Hamiltonian acting on it gives

$$\hat{H}|0\rangle = \hbar\omega\left(0 + \tfrac{n}{2}\right)|0\rangle = \frac{n}{2}\hbar\omega\,|0\rangle$$

Therefore, the **ground-state energy** is

$$E_0 = \frac{n}{2}\hbar\omega$$

(4) Energy spectrum

Starting from $|0\rangle$, we can build a general **number state**

$$|n_1, n_2, \dots, n_n\rangle = \prod_{i=1}^{n} \frac{(a_i^\dagger)^{n_i}}{\sqrt{n_i!}} |0\rangle$$

where $n_i \in \mathbb{N}_0$ counts how many quanta occupy mode i.

Because $[a_i^\dagger a_i, a_j^\dagger a_j] = 0$, these states are simultaneous eigenstates of all number operators:

$$a_i^\dagger a_i |n_1, \dots, n_n\rangle = n_i |n_1, \dots, n_n\rangle$$

and thus thet are also energy eigenstates.
Operating with $\hat{H}$:

$$\hat{H}|n_1, \dots, n_n\rangle = \hbar\omega \sum_{i=1}^{n} \left(n_i + \tfrac{1}{2}\right)|n_1, \dots, n_n\rangle$$

read the energy eigenvalues

$$E_{n_1,\dots,n_n} = \hbar\omega\left(\sum_{i=1}^{n} n_i + \tfrac{n}{2}\right) = \hbar\omega\left(N_{\text{tot}} + \tfrac{n}{2}\right)$$

where $N_{\text{tot}} = n_1 + n_2 + \cdots + n_n$ is the **total excitation number**.

(5) Degeneracy counting

For fixed $N_{\text{tot}} = N$, the degeneracy equals the number of integer partitions of N into n nonnegative integers:

$$g_N = \binom{N+n-1}{n-1}$$

III. Bilinear Operators $a_i^\dagger a_j$

We define the bilinear operators

$$E_{ij} = a_i^\dagger a_j$$

(1) Verify that bilinear operators $a_i^\dagger a_j$ commutates with Hamiltonian

Compute their commutator with $\hat{H}$:

$$[\hat{H}, E_{ij}] = \hbar\omega \sum_k [a_k^\dagger a_k, a_i^\dagger a_j] = \hbar\omega\big([a_i^\dagger a_i, a_i^\dagger a_j] + [a_j^\dagger a_j, a_i^\dagger a_j]\big) = 0$$

Thus all E_{ij} commute with $\hat{H}$.

(2) A quick reminder of Lie algebra

A **Lie algebra** is a vector space $\mathfrak{g}$ equipped with a binary operation (the **Lie bracket** $[\cdot,\cdot]$) satisfying three properties:

1. **Bilinearity:**
   $[aX + bY, Z] = a[X, Z] + b[Y, Z]$, etc.
2. **Antisymmetry:**
   $[X, Y] = -[Y, X]$.
3. **Jacobi identity:**
   $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.
   When these are satisfied, $(\mathfrak{g}, [\cdot,\cdot])$ is called a *Lie algebra*.

(3) $(\{E_{ij}\}, [\cdot,\cdot])$ indeed form a Lie algebra

We use the product rule:

$$[E_{ij}, E_{kl}] = [a_i^\dagger a_j, a_k^\dagger a_l] = a_i^\dagger [a_j, a_k^\dagger] a_l - a_k^\dagger [a_l, a_i^\dagger] a_j$$

using $[a_j, a_k^\dagger] = \delta_{jk}$ and $[a_l, a_i^\dagger] = \delta_{li}$, we get

$$[E_{ij}, E_{kl}] = \delta_{jk} a_i^\dagger a_l - \delta_{li} a_k^\dagger a_j = \delta_{jk} E_{il} - \delta_{il} E_{kj}$$

and thus E_{ij} generate the **Lie algebra** $\mathfrak{u}(n)$ under commutation, since:

- **Linearity:** obvious from the linearity of the commutator.
- **Antisymmetry:** since $[E_{ij}, E_{kl}] = -[E_{kl}, E_{ij}]$, this holds.
- **Jacobi identity:** holds automatically for all commutators of operators, because the commutator bracket in operator algebra always satisfies Jacobi.

(4) Cartan subalgebra

The **Cartan subalgebra** is the maximal abelian subset of this Lie algebra.
Since we have shown that:

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$$

a natural choice on elements of the subset is the diagonal set:

$$H_i = E_{ii} = a_i^\dagger a_i, \quad i = 1, \ldots, n$$

since $[H_i, H_j] = 0$ ensures it's abelian. These are just the number operators we defined in (4) Energy spectrum, each measures the occupation number in mode i.

The simultaneous eigenstates of all H_i are the number states $|n_1, \ldots, n_n\rangle$ with

$$H_i|n_1, \ldots, n_n\rangle = n_i|n_1, \ldots, n_n\rangle,$$

which are also energy eigenstates of $\hat{H}$.

IV. Takeaways

- The n-dimensional oscillator is **a set of n independent 1D oscillators**.
- The **Hamiltonian** depends only on the total number operator $N = \sum_i a_i^\dagger a_i$.
- The **symmetry group** of the degeneracies is $U(n)$, generated by $a_i^\dagger a_j$.
- Each energy level $E_N = \hbar\omega(N + n/2)$ forms a representation of $U(n)$ of dimension $\binom{N+n-1}{n-1}$.

Exercise 6

At the end of the previous lecture we introduced the harmonic oscillators $a_r(\vec{k})$ and $a_r^\dagger(\vec{k})$, where $r \in \{0, 1, 2, 3\}$ and $\vec{k} \in (2\pi/L)\mathbb{Z}^3$, satisfying

$$[a_r(\vec{k}), a_s(\vec{k}')] = [a_r^\dagger(\vec{k}), a_s^\dagger(\vec{k}')] = 0,$$

and

$$[a_r(\vec{k}), a_s^\dagger(\vec{k}')] = \zeta_r \, \delta_{rs} \, \delta_{\vec{k},\vec{k}'}, \quad \zeta_{1,2,3} = +1, \ \zeta_0 = -1$$

Define the Hermitian operator

$$N(\vec{k}) = \sum_{r=0}^{3} \zeta_r \, a_r^\dagger(\vec{k}) \, a_r(\vec{k}),$$

and show that

$$[N(\vec{k}'), a_r^\dagger(\vec{k})] = \delta_{\vec{k}',\vec{k}}\, a_r^\dagger(\vec{k}), \qquad [N(\vec{k}'), a_r(\vec{k})] = -\delta_{\vec{k}',\vec{k}}\, a_r(\vec{k}).$$

Then reread the last part of the lecture where the ground state and first excited states were analyzed.

Solution 6

I. Commutator $[N(\vec{k}'), a_r^\dagger(\vec{k})]$

(1) Use linearity of commutator

Start from linearity of commutator:

$$[N(\vec{k}'), a_r^\dagger(\vec{k})] = \sum_{s=0}^{3} \zeta_s \left[a_s^\dagger(\vec{k}')\, a_s(\vec{k}'),\ a_r^\dagger(\vec{k})\right]$$

(2) Use Lebniz rule of commutator

(For more detailed proof, refer to Lecture 1Y Commutator as a Lie Derivative, online Lecture 1X : Topological Groups, Lie Groups and Lie Algebra in Quantum Physics (revised) - Physics Reserved Labour)
Use the Leibniz rule $[AB, C] = A[B, C] + [A, C]B$:

$$= \sum_s \zeta_s \Big(a_s^\dagger(\vec{k}')[\, a_s(\vec{k}'), a_r^\dagger(\vec{k})] + [a_s^\dagger(\vec{k}'), a_r^\dagger(\vec{k})]\, a_s(\vec{k}') \Big)$$

- The second bracket vanishes because $[a_s^\dagger, a_r^\dagger] = 0$.
- For the first:

$$[\, a_s(\vec{k}'), a_r^\dagger(\vec{k})] = \zeta_s\, \delta_{sr}\, \delta_{\vec{k}',\vec{k}}$$

(3) Sum over indices

$$[N(\vec{k}'), a_r^\dagger(\vec{k})] = \sum_s \zeta_s\, a_s^\dagger(\vec{k}') \left(\zeta_s\, \delta_{sr}\, \delta_{\vec{k}',\vec{k}}\right) = \delta_{\vec{k}',\vec{k}}\, a_r^\dagger(\vec{k})$$

(We used $\zeta_s^2 = 1$.)

II. Commutator $[N(\vec{k}'), a_r(\vec{k})]$

Similarly,

$$[N(\vec{k}'), a_r(\vec{k})] = \sum_s \zeta_s \, [a_s^\dagger(\vec{k}') a_s(\vec{k}'), a_r(\vec{k})]$$

which, computed by Leibniz rule, is:

$$= \sum_s \zeta_s \Big(a_s^\dagger(\vec{k}')[a_s(\vec{k}'), a_r(\vec{k})] + [a_s^\dagger(\vec{k}'), a_r(\vec{k})] a_s(\vec{k}') \Big)$$

Again the first bracket vanishes because $[a_s, a_r] = 0$.
For the second:

$$[a_s^\dagger(\vec{k}'), a_r(\vec{k})] = -[a_r(\vec{k}), a_s^\dagger(\vec{k}')] = -\, \zeta_r \, \delta_{rs} \, \delta_{\vec{k},\vec{k}'}$$

so eventually we have

$$[N(\vec{k}'), a_r(\vec{k})] = \sum_s \zeta_s \big(-\, \zeta_r \, \delta_{rs} \, \delta_{\vec{k},\vec{k}'} \big) \, a_s(\vec{k}') = -\, \delta_{\vec{k}',\vec{k}} \, a_r(\vec{k})$$

Exercise 7

After introducing periodic boundary conditions $x \simeq x + L$, $y \simeq y + L$, $z \simeq z + L$, we obtained the solution

$$A^\mu(x) = A_+^\mu(x) + A_-^\mu(x),$$

with

$$A_+^\mu(x) = \sum_{r=0}^{3} \sum_{\vec{k} \in \frac{2\pi}{L}\mathbb{Z}^3} \sqrt{\frac{c^2}{2V\omega_{\vec{k}}}} \, \varepsilon_r^\mu(\vec{k}) \, a_r(\vec{k}) \, e^{-ik\cdot x},$$

and

$$A_-^\mu(x) = \sum_{r=0}^{3} \sum_{\vec{k} \in \frac{2\pi}{L}\mathbb{Z}^3} \sqrt{\frac{c^2}{2V\omega_{\vec{k}}}} \, \varepsilon_r^\mu(\vec{k}) \, a_r^\dagger(\vec{k}) \, e^{+ik\cdot x},$$

where $V = L^3$ and $\omega_{\vec{k}} = c|\vec{k}|$.
The oscillators satisfy the commutation relations given above.

1. Show that to take the limit $L \to \infty$, we must rescale

$$a_r(\vec{k}) \to \tilde{a}_r(\vec{k}) = \sqrt{\frac{V}{(2\pi)^3}}\, a_r(\vec{k}), \qquad \tilde{a}_r^\dagger(\vec{k}) = (\tilde{a}_r(\vec{k}))^\dagger.$$

   In this limit, give the commutation relations satisfied by $\tilde{a}_r(\vec{k})$ and $\tilde{a}_r^\dagger(\vec{k})$.
2. Give the expressions for $A_+^\mu(x)$ and $A_-^\mu(x)$ in that limit.

Solution 7

I. Necessity of Rescaling

(0) Setup

We start from the finite-volume (periodic box) mode solution:

$$A^\mu(x) = A_+^\mu(x) + A_-^\mu(x),$$

and ladder operators obeying:

$$[a_r(\vec{k}), a_s(\vec{k}')] = [a_r^\dagger(\vec{k}), a_s^\dagger(\vec{k}')] = 0, \qquad [a_r(\vec{k}), a_s^\dagger(\vec{k}')] = \zeta_r\, \delta_{rs}\, \delta_{\vec{k},\vec{k}'},$$

with $\zeta_{1,2,3} = +1$ and $\zeta_0 = -1$.
Our goal is to take $L \to \infty$ (continuum limit) **carefully** so that:

- sums over discrete $\vec{k}$ become integrals over $\mathbb{R}^3$,
- Kronecker deltas become Dirac deltas,
- field operators remain **finite** and have the **standard continuum commutators**.

(1) What if we use the finite/discrete form directly?

In a periodic box, the allowed momenta are $\vec{k} = \frac{2\pi}{L}\vec{n}$ with $\vec{n} \in \mathbb{Z}^3$.
The sum over $\vec{k}$ is a Riemann sum with k-space cell volume

$$(\Delta k)^3 = \left(\frac{2\pi}{L}\right)^3 = \frac{(2\pi)^3}{V}$$

- Hence, for any "well-defined" function f, when we take the infinite limit, we'll have:

$$\sum_{\vec{k}} f(\vec{k}) \xrightarrow[L\to\infty]{} \frac{V}{(2\pi)^3} \int d^3k\, f(\vec{k})$$

- Likewise, the Kronecker delta in momentum space converts to a Dirac delta with the **inverse** measure:

$$\delta_{\vec{k},\vec{k}'} \xrightarrow[L\to\infty]{} \frac{(2\pi)^3}{V}\, \delta^{(3)}(\vec{k}-\vec{k}')$$

Which means if we do nothing to the operators $a_r(\vec{k})$, then:

- the sum $\sum_{\vec{k}}$ brings in a factor V,
- the field prefactor $\sqrt{c^2/(2V\omega_{\vec{k}})}$ brings a factor $V^{-1/2}$,

so the **overall scaling** would be $\propto \sqrt{V}$, which **diverges** as $L \to \infty$. Therefore, a compensating rescaling of the oscillators is necessary to keep $A^\mu(x)$ finite.

(2) Construct the rescaling

To cancel the $\propto \sqrt{V}$ overall divergence, the natural solution is to impose that:

$$a_r(\vec{k}) = \text{SomeFatcor} \cdot \sqrt{\frac{1}{V}} \tilde{a}_r(\vec{k})$$

and to preserve normalization, we choose:

$$\tilde{a}_r(\vec{k}) \equiv \sqrt{\frac{V}{(2\pi)^3}}\, a_r(\vec{k}) \quad \Longleftrightarrow \quad a_r(\vec{k}) = \sqrt{\frac{(2\pi)^3}{V}}\, \tilde{a}_r(\vec{k}).$$

So that **all powers of V cancel exactly** and normalization preserved:

$$\frac{V}{(2\pi)^3} \times \frac{1}{\sqrt{V}} \times \sqrt{\frac{(2\pi)^3}{V}} \;=\; 1$$

the field is **finite** and has the standard continuum normalization.

(3) Verify the rescaled operator

Check the commutator:

$$[\tilde{a}_r(\vec{k}), \tilde{a}_s^\dagger(\vec{k}')] = \frac{V}{(2\pi)^3}\,[a_r(\vec{k}), a_s^\dagger(\vec{k}')] = \frac{V}{(2\pi)^3}\,\zeta_r\,\delta_{rs}\,\delta_{\vec{k},\vec{k}'} \xrightarrow[L\to\infty]{} \zeta_r\,\delta_{rs}\,(2\pi)^3\,\delta^{(3)}(\vec{k}-\vec{k}')$$

where we used $\delta_{\vec{k},\vec{k}'} \to \frac{(2\pi)^3}{V}\,\delta^{(3)}(\vec{k}-\vec{k}')$.

Similarly,

$$[\tilde{a}_r(\vec{k}), \tilde{a}_s(\vec{k}')] = [\tilde{a}_r^\dagger(\vec{k}), \tilde{a}_s^\dagger(\vec{k}')] = 0.$$

Therefore, in the continuum limit

$$\boxed{[\tilde{a}_r(\vec{k}), \tilde{a}_s^\dagger(\vec{k}')] = \zeta_r\,\delta_{rs}\,(2\pi)^3\,\delta^{(3)}(\vec{k}-\vec{k}'), \qquad [\tilde{a},\tilde{a}] = [\tilde{a}^\dagger,\tilde{a}^\dagger] = 0.}$$

II. Solutions in Continuum Limit

(1) A quick reminder

Use:

$$\boxed{\sum_{\vec{k}} \to \frac{V}{(2\pi)^3}\int d^3k}$$

and

$$a_r(\vec{k}) = \sqrt{\frac{(2\pi)^3}{V}}\,\tilde{a}_r(\vec{k})$$

(2) Substitue and give results

For $A_+^\mu(x)$:

$$A_+^\mu(x) \xrightarrow[L\to\infty]{} \sum_r \frac{V}{(2\pi)^3}\int d^3k\,\sqrt{\frac{c^2}{2V\omega_{\vec{k}}}}\,\varepsilon_r^\mu(\vec{k})\,\left(\sqrt{\frac{(2\pi)^3}{V}}\,\tilde{a}_r(\vec{k})\right)\,e^{-ik\cdot x}$$

and this is equal to

$$= \sum_r \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{c^2}{2\omega_{\vec{k}}}}\, \varepsilon_r^\mu(\vec{k})\, \tilde{a}_r(\vec{k})\, e^{-ik\cdot x}$$

In the same manner, $A_-^\mu(x)$ gives

$$A_-^\mu(x) = \sum_r \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{c^2}{2\omega_{\vec{k}}}}\, \varepsilon_r^\mu(\vec{k})\, \tilde{a}_r^\dagger(\vec{k})\, e^{+ik\cdot x}$$

Thus, in the continuum limit, we conclude:

$$\boxed{\begin{aligned} A_+^\mu(x) &= \sum_{r=0}^{3} \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{c^2}{2\omega_{\vec{k}}}}\, \varepsilon_r^\mu(\vec{k})\, \tilde{a}_r(\vec{k})\, e^{-ik\cdot x}, \\ A_-^\mu(x) &= \sum_{r=0}^{3} \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{c^2}{2\omega_{\vec{k}}}}\, \varepsilon_r^\mu(\vec{k})\, \tilde{a}_r^\dagger(\vec{k})\, e^{+ik\cdot x}. \end{aligned}}$$

Exercise 8

While we are presently developing the material in chapter 5 of Mandl & Shaw, it is most useful to read **chapter 3** (scalar fields) alongside it.
Because of their simple Lorentz-transformation properties and the absence of gauge symmetry, scalar fields are in many respects simpler than massless vector fields while still illustrating many important concepts.
At this point it is **highly advisable** to carefully study sections 3.1 and 3.2.

Exercise 9

Together with this assignment you received a file **Poincaré.pdf** developing the representation theory of the Poincaré group.
Study it carefully!

You may take your time, but you should have a grasp of it by the end of this calendar year.