Lecture 1S

A Quick Reminder of Elementary Hamiltonian Mechanics

1. Phase Space

I. Phase Space $\mathcal{P}$ is a smooth $2n$-dimensional Manifold

Phase space $\mathcal{P}$
The basic object in classical mechanics (CM) is the **phase space** $\mathcal{P}$.

- $\mathcal{P}$ is a smooth manifold of even dimension $d = 2n$.
- A point in phase space is

$$\xi \in \mathcal{P}$$

  corresponds to a **pure state** of a physical system.
  Thus, $\mathcal{P}$ represents the space of all possible **pure states** of a system.

II. Phase Space $\mathcal{P}$ is a Simplectic Manifold

Beyond being a smooth $2n$-dimensional manifold, the phase space $\mathcal{P}$ carries an additional geometric structure: a **symplectic form**.

(1) Symplectic form field

A **symplectic form** on $\mathcal{P}$ is a smooth 2-form field

$$\omega \in \Omega^2(\mathcal{P})$$

that satisfies two conditions:

1. **Closedness:** $d\omega = 0$
2. **Non-degeneracy:** For every $\xi \in \mathcal{P}$ and nonzero $v \in T_\xi\mathcal{P}$,
   there exists some $w \in T_\xi\mathcal{P}$ such that $\omega(v, w) \neq 0$.

In canonical local coordinates (q_i, p_i) on $\mathcal{P}$, the standard expression is

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

(1') A smooth manifold equipped with a symplectic form field is called a simplectic manifold

A pair $(\mathcal{P}, \omega)$, where $\mathcal{P}$ is a smooth manifold and ω a symplectic form on it, is called a **symplectic manifold**.
The symplectic form provides a coordinate-free way to encode the canonical structure of phase space.

(2) Darboux's theorem and canonical coordinates

Darboux's theorem states that given a **symplectic manifold** $\mathcal{P}$, around every point $\xi_0 \in \mathcal{P}$, there exists a local coordinate chart

$$(q_1, \dots, q_n, p_1, \dots, p_n)$$

such that

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

These local coordinates are called **canonical coordinates** (or **Darboux coordinates**).

(2') Symplectic structure give rise to a canonical coordinates

In other words, given a **symplectic manifold**, we can always find a (local) coordinate chart, such that the symplectic form field takes a simple coorinate form, and we call such coordinates **canonical coordinates**.

Later we shall see:

> The **symplectic form** is the coordinate-free essence of Hamiltonian geometry;
> While **canonical coordinates** are local charts in which this structure takes its simplest form.

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2. States

In Hamiltonian mechanics, a **state** represents our knowledge (complete or incomplete) about where the system lies in phase space.
Mathematically, a state is described by a **distribution function** on the phase space $\mathcal{P}$.

I. General State ρ

A **general state** (or **statistical state**) is a smooth, non-negative, normalized function

$$\rho : \mathcal{P} \to \mathbb{R}_{\geq 0},$$

satisfying

$$\int_{\mathcal{P}} \rho(q,p)\, d\mu_\omega = 1.$$

were $d\mu_\omega$ is the **Liouville measure** on $\mathcal{P}$, defined from the symplectic form:

$$d\mu_\omega = \frac{\omega^{\wedge n}}{n!}.$$

- $\rho(q,p)$ represents the **probability density** of finding the system near point (q,p) in phase space.
- In **canonical coordinates** $(q_1,\ldots,q_n,p_1,\ldots,p_n)$, the Liouville measure takes the coordinate expression

  $$d\mu_\omega = dq_1 \wedge \cdots \wedge dq_n \wedge dp_1 \wedge \cdots \wedge dp_n.$$

  Thus, the normalization condition becomes

  $$\int_{\mathbb{R}^{2n}} \rho(q,p)\, dq_1 \cdots dq_n\, dp_1 \cdots dp_n = 1$$

Thus, ρ encodes both deterministic and statistical information about the system.

II. Pure State

A **pure state** is an extremal case of ρ that represents complete knowledge of the system's position in phase space.

Mathematically, it is a **Dirac distribution** supported on a single point:

$$\rho_\xi(q,p) = \delta(q-q_0)\,\delta(p-p_0),$$

where $\xi_0 = (q_0, p_0) \in \mathcal{P}$ is the phase-space point describing the system. **In practice**, many abuse the notation of phase-space point ξ_0 to represent this distribution directly.

Hence:

- **Pure states** $\leftrightarrow$ points in $\mathcal{P}$ (deterministic trajectories).
- **Mixed states** $\leftrightarrow$ probability distributions on $\mathcal{P}$ (ensembles).

III. Convex Geometry

The set of all normalized, non-negative distribution functions (which represent all general states of system)

$$\mathcal{S} = \left\{ \rho \in C^\infty(\mathcal{P}) \,\middle|\, \rho \ge 0,\ \int \rho\, d\mu_\omega = 1 \right\}$$

forms a **convex set**.
By saying $\mathcal{S}$ is a convex set, we mean: if $\rho_1, \rho_2 \in \mathcal{S}$ and $0 \le \lambda \le 1$, then

$$\rho = \lambda\rho_1 + (1-\lambda)\rho_2 \in \mathcal{S}.$$

One can also prove that:

- **Pure states** are the **extreme points** of this convex set:
  they cannot be written as convex combinations of other states.
- **Mixed states** are interior points: convex mixtures of pure states.

Quantum Mechanics Counterpart

This **convex geometry** of states in classical mechanics mirrors that of quantum mechanics, where the set of density matrices also forms a convex set with pure states as its extremal elements.

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3. Observables

In Hamiltonian mechanics, an **observable** represents any measurable physical quantity of a system — for example, position, momentum, energy, or angular momentum.
Mathematically, observables are real-valued smooth functions on the phase space $\mathcal{P}$.

I. Observable: Smooth Function on Phase Manifold

An **observable** is a smooth function

$$f : \mathcal{P} \to \mathbb{R}, \qquad f \in C^\infty(\mathcal{P}).$$

In canonical coordinates (q_i, p_i), this means

$$f = f(q_1, \ldots, q_n,\, p_1, \ldots, p_n).$$

Each observable assigns to every pure state $\xi = (q, p)$ a real number $f(\xi)$ — the value of that quantity when the system is in state ξ.

II. Algebraic Structure of Observables and Poisson Braket

The set $C^\infty(\mathcal{P})$ of smooth observables carries two compatible algebraic structures:

1. **Pointwise multiplication**

$$(f, g) \;\mapsto\; (fg)(\xi) = f(\xi)g(\xi).$$

2. **Poisson bracket** induced by the symplectic form ω:
   Given symplectic form field ω, every smooth function f defines a **Hamiltonian vector field** X_f by

$$\iota_{X_f}\omega = df.$$

   The **Poisson bracket** between two observables f and g is defined as

$$\{f, g\} = \omega(X_f, X_g).$$

   In canonical coordinates, this becomes

\{f,g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial q_i} \frac{\partial

- \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}
\right)$$

(1) Symlectic structure naturally leads to Poisson braket for smooth function on symplectic manifold

Given the symplectic form field ω, every smooth function $f \in C^\infty(\mathcal{P})$ defines a unique **Hamiltonian vector field** X_f through:

$$\iota_{X_f} \omega = df.$$

The **Poisson bracket** of two observables $f, g \in C^\infty(\mathcal{P})$ is then defined by

$$\{f, g\} = \omega(X_f, X_g).$$

This definition uses only the symplectic structure and is therefore coordinate-free.

(2) Properties of the Poisson Bracket

The Poisson bracket endows $C^\infty(\mathcal{P})$ with the structure of a **Poisson algebra**, satisfying:

- **Bilinearity**

$$\{af + bg,\ h\} = a\{f, h\} + b\{g, h\}, \qquad a, b \in \mathbb{R}.$$

- **Antisymmetry**

$$\{f, g\} = -\{g, f\}.$$

- **Jacobi identity**

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

- **Leibniz rule (derivation property)**

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

These properties ensure that the Poisson bracket acts as a **Lie bracket** on observables while remaining compatible with pointwise multiplication.
They are the classical analog of the commutator algebra in quantum mechanics.

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4. Dynamics

Time evolution in Hamiltonian mechanics is generated by a distinguished observable — the **Hamiltonian function** $H : \mathcal{P} \to \mathbb{R}$, which represents the total energy of the system.

The symplectic form ω allows H to define a vector field on $\mathcal{P}$, whose integral curves describe the motion of the system in phase space.

I. Hamiltonian Vector Field

(1) Geometric definition

Given a Hamiltonian function $H \in C^\infty(\mathcal{P})$, the associated **Hamiltonian vector field**

$$X_H \in \mathfrak{X}(\mathcal{P})$$

is defined implicitly by

$$\iota_{X_H}\omega = dH.$$

- ι_{X_H} denotes interior contraction of ω with X_H.
- Non-degeneracy of ω ensures the existence and uniqueness of X_H.

(2) Canonical-coordinatewise

In **canonical coordinates** (q_i, p_i) with

$$\omega = \sum_i dq_i \wedge dp_i,$$

the **Hamiltonian vector field** can be explicitly defined as: $$X*H*
= *sum*{i=1}^n
\left(
\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}

- \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}
\right).$$

II. Hamilton’s Equations

Let a **trajectory** in phase space be a smooth curve

$$\xi(t) = (q(t), p(t)) \in \mathcal{P}.$$

Time evolution is determined by the **Hamiltonian function**

$$H : \mathcal{P} \to \mathbb{R},$$

which specifies the total energy of the system.

(1) Postulate: physical motion of system satisfies the condition " is an **integral curve** of the Hamiltonian vector field X_H:"

The motion of the system is described by the condition that $\xi(t)$ is an **integral curve** of the Hamiltonian vector field X_H:

$$\dot{\xi}(t) = X_H(\xi(t)).$$

This means that, at every instant, the velocity vector of the trajectory in phase space equals the value of X_H at that point.

(2) Result: Halmilton's equation

In principle, we may call the equation $\dot{\xi}(t) = X_H(\xi(t))$. above the Hamilton's equation of motion; but **in practice**, since we work on **canonical coordinates** (q_i, p_i) to descrive the state of system: recalling that

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right),$$

this yields the familiar canonical **Hamilton’s equations of motion**:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

(3) Define: Hamiltonian Flow and Canonical Transformations

Let $\Phi_t : \mathcal{P} \to \mathcal{P}$ denote the **Hamiltonian flow** generated by X_H, defined by

$$\frac{d}{dt}\Phi_t(\xi_0) = X_H(\Phi_t(\xi_0)), \qquad \Phi_0 = \mathrm{id}.$$

We can show that each Φ_t is a **diffeomorphism** of $\mathcal{P}$ that preserves the symplectic form:

$$\Phi_t^* \omega = \omega$$

Such transformations are called **canonical transformations** or **symplectomorphisms**.

Hence, time evolution in Hamiltonian mechanics is realized as a **one-parameter family of canonical transformations** of phase space.

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(4) Poisson Bracket Form of Hamilton's Equations

Let $f(q,p)$ be any smooth observable on $\mathcal{P}$.
Its total time derivative along the trajectory $\xi(t)$ is

$$\frac{df}{dt} = \sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right).$$

Substituting Hamilton's equations gives

$$\frac{df}{dt} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = \{f, H\}$$

Thus, **Hamilton's equations** can be equivalently expressed as the **Poisson-bracket evolution law**:

$$\boxed{\frac{df}{dt} = \{f, H\}.}$$

(5) Recover the canonical Hamilton's equations from Poisson Braket Hamilton's equation

In particular, for the canonical coordinates themselves:

$$\dot{q}_i = \{q_i, H\}, \qquad \dot{p}_i = \{p_i, H\}.$$

This shows that the Poisson bracket is not merely an algebraic structure — it encodes the **dynamical generator of time evolution**.

III. Liouville Equation and Phase Flow

When the system's state is a **statistical distribution** $\rho(q,p,t)$ on phase space, time evolution is governed by the **Liouville equation**.

If we imagine following the flow (governed by the Hamiltonian) of phase-space points — like watching dye particles move in an incompressible fluid — the probability density "attached" to each moving point doesn't change:

$$\frac{d\rho}{dt} = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + X_H(\rho) = 0$$

In canonical coordinates, using the expression for X_H, this becomes$$\frac{\partial
\rho}{\partial t}
= -\sum_{i=1}^n
\left(
\frac{\partial H}{\partial p_i}\frac{\partial \rho}{\partial q_i}

- \frac{\partial H}{\partial q_i}\frac{\partial \rho}{\partial p_i}
  \right)

$$or equivalently,$$

\frac{\partial \rho}{\partial t} = {H,\rho}.$$

Thus, the Liouville equation expresses the **conservation of probability density** along the Hamiltonian flow.

The Hamiltonian flow preserves both the symplectic form and the Liouville measure:

$$\Phi_t^* \omega = \omega, \qquad \Phi_t^* d\mu_\omega = d\mu_\omega,$$

ensuring that the volume of any region in phase space remains invariant under time evolution — this is **Liouville's theorem**.