Lecture 1X Lie Group and Lie Algebra

Topological Group, Lie Group and Lie Algebra, with Applications in Quantum Physics

> This is just a **translation** of another note 000 从拓扑群到李群 from Group Theory. I do made some **modifications** to fi it in Quantum Theory context. This note assumes reader is completely ignorant of Lie algebras

We first clarify our **onventions**:

- Hilbert-space inner product is linear in the **second** slot.
- Spacetime metric: choose $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$ unless stated.
- Commutator: $[A, B] = AB - BA$.
- In physics, Lie-algebra generators are usually taken **Hermitian**; the mathematician's anti-Hermitian basis differs by factors of i.

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1. From Topological Groups to Lie Groups

I. Groups

A **group** $(G, \cdot)$ is a set equipped with a binary operation satisfying:

1. **Closure:** For any $g, h \in G$, $gh \in G$.
2. **Associativity:** $(gh)k = g(hk)$.
3. **Identity:** There exists $e \in G$ such that $ge = eg = g$.
4. **Inverse:** For every $g \in G$, there exists g^{-1} with $gg^{-1} = g^{-1}g = e$.
   A group may be discrete, finite, or infinite. Its structure is purely algebraic — no geometry or topology is assumed yet.

II. Topological Groups

Topological Space

(1) Definition

Let X be a nonempty set.
If there is a specified collection of subsets $\mathcal{T} \subset \mathcal{P}(X)$ (where $\mathcal{P}(X)$ is the power set of X) satisfying the following three axioms, then the pair $(X, \mathcal{T})$ is called a **topological space**, $\mathcal{T}$ is called a **topology** on X, and its elements are called **open (sub)set** of X :

1. **The empty set and the whole set are open:**

$$\in \mathcal{T}, \quad X \in \mathcal{T}.$$

2. **Arbitrary unions of open sets are open:**
If $\{U_i\}_{i \in I} \subset \mathcal{T}$, then

$$_{i \in I} U_i \in \mathcal{T}.$$

3. **Finite intersections of open sets are open:**
If $U_1, \ , U_n \in \mathcal{T}$, then

$$^{n}_{j=1} U_j \in \mathcal{T}.$$

Concepts such as *continuity*, *convergence*, *connectedness*, and *compactness* are all defined relative to the chosen topology $\mathcal{T}$.

(2) Continuity of Maps between Topological Spaces

A map $f : X \to Y$ between two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is said to be **continuous** if:
for every open set $\ \in \mathcal{T}_Y$, its **preimage** $f^{-1}()$ is open in X:

$$\in \mathcal{T}_Y \quad f^{-1}() \in \mathcal{T}_X.$$

From a categorical perspective, continuous maps are those that *preserve the topological structure*, just as group **homomorphisms** preserve group structure and linear maps preserve linear structure.

(2') Homomorphisms between topological spaces = continuous maps

Formally, let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces.
A map

$$f : X \to Y$$

is called a **topological homomorphism** (or simply a **continuous map**) if it satisfies:

$$\in \mathcal{T}_Y \quad f^{-1}() \in \mathcal{T}_X.$$

That is, the preimage of every open set in Y is open in X.

In category-theoretic language:

- Objects: topological spaces $(X, \mathcal{T}_X)$
- Morphisms (homomorphisms): continuous maps $f : X \to Y$

(3) Homeomorphism = bijection + continuous + inversely continuous

Let $(X, _X)$ and $(Y, _Y)$ be topological spaces.
A map $f : X \to Y$ is called a **homeomorphism** if:

1. f is a **bijection** (one-to-one and onto).
2. f is **continuous**.
3. The inverse map $f^{-1} : Y \to X$ is also **continuous**.
   If such an f exists, we say that X and Y are **homeomorphic**, meaning they are *topologically equivalent*.

Topological Groups

A **topological group** is a mathematical structure that combines two compatible ingredients:

1. **Group structure:** it allows algebraic operations — multiplication and inversion.
2. **Topological structure:** it allows notions of continuity, limits, and convergence.

(1) Formal definition

Formally, a topological group is a group G that is also a topological space, such that:

- The **multiplication map**

$$\mu : G \times G \to G, \qquad (g,h) \mapsto gh$$

  is **continuous** with respect to the product topology on $G \times G$.
- The **inversion map**

$$\iota : G \to G, \qquad g \mapsto g^{-1}$$

  is also **continuous**.
  These two conditions ensure that the algebraic operations of the group are *compatible* with the topology — small changes in the elements of G lead to small changes in their product and inverse.

(2) Intuitive meaning

A topological group provides a framework where one can discuss **continuous deformations** of group elements. This means we can talk about *paths*, *connected components*, and *limits* inside a group while preserving its algebraic structure.
In physics, this is crucial for describing **continuous symmetries**, where group parameters can vary smoothly.

Typical examples

1. **Additive group of real numbers:**
   $(\mathbb{R}, +)$ is a topological group with the usual topology; both addition and negation are continuous.
2. **The circle group:**
   $S^1 = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$,
   with multiplication of complex numbers. It represents the simplest compact Lie group.
3. **General linear group:**
   $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$,
   endowed with the subspace topology inherited from $\mathbb{R}^{n^2}$.
   Matrix multiplication and inversion are continuous functions of the entries.

4. **Discrete groups:**
   Any group endowed with the discrete topology is trivially a topological group, since all maps are continuous.

III. Motivation for Lie Groups

Topological groups can describe continuity, but not differentiability.
In physics, we need *smooth* transformations — infinitesimal symmetries that can be generated by derivatives.
This leads to **Lie groups**, which possess a differentiable structure compatible with the group operation.

IV. Lie Group

A **Lie group** is a mathematical structure that unifies **smooth geometry** and **group algebra**.
It is a group G that is also a **smooth manifold**, such that the group operations (prodect, inversion) are **smooth maps** in the sense of differential geometry.

(1) Formal definition

A **Lie group** is a pair $(G, \mathcal{A})$, where G is a set and $\mathcal{A}$ is a **smooth manifold** structure on G, satisfying:

- The **multiplication map**

  $$\mu : G \times G \to G, \qquad (g, h) \mapsto gh$$

  is smooth (C^∞) with respect to the product manifold structure on $G \times G$.
- The **inversion map**

  $$\iota : G \to G, \qquad g \mapsto g^{-1}$$

  is also smooth.

In other words, both operations of **composing** and **inverting** of group elements are *differentiable operations*. These two smoothness conditions ensure that the algebraic and differential structures are **compatible**.

> Notice: we require that the manifold to be **smooth** instead of only **topological** to clarify that there is a compactible **differential structure** defined on this space.

(2) Direct consequence as a smooth manifold

Hence, one can study both **global symmetries** (through the group structure) and **local/infinitesimal symmetries** (through calculus on the manifold).
Every path through the identity element of G corresponds to a **continuous symmetry transformation**; taking derivatives along such paths gives rise to **infinitesimal generators**, forming the **Lie algebra** of G.

(3) Examples

1. **Abelian Lie group:**
   $(\mathbb{R}^n, +)$ — addition and inversion are smooth linear maps.
2. **Matrix Lie groups:**
   Subgroups of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ defined by smooth constraints, e.g.
   - $O(n) = \{A \in GL(n, \mathbb{R}) \mid A^T A = I\}$
   - $SO(3)$ — rotation group in 3D
   - $SU(2)$ — special unitary group of degree 2, double cover of $SO(3)$
3. **Circle group:**
   $S^1 = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$
   with complex multiplication, a compact 1D Lie group.
4. **Discrete groups as zero-dimensional Lie groups:**
   If G carries the discrete topology, then every map is smooth by definition, so any discrete group is trivially a 0-dimensional Lie group.

(4) Why Lie groups matter

Lie groups are the **natural mathematical language of continuous symmetries**:

- In **classical mechanics**, they describe canonical transformations (e.g., rotations, boosts).
- In **quantum mechanics and QFT**, they govern unitary transformations on Hilbert space.

- Their tangent spaces at the identity define **Lie algebras**, whose elements correspond to **infinitesimal generators** of these symmetries.
  Thus, Lie groups provide the bridge between **geometry (manifolds)** and **symmetry (group actions)**, forming the foundation for the algebraic structures used in modern physics.

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2. From Lie Group to Lie Algebra

I. Motivation

A **Lie group** G encodes *global*, finite symmetry transformations — for example, finite rotations, boosts, or translations.
However, in both mathematics and physics, we are often more interested in the **infinitesimal** structure near the identity element $e \in G$:

- In calculus, local derivatives describe how a function behaves near a point.
- In symmetry theory, local derivatives describe how an infinitesimal transformation acts.

To describe such infinitesimal structure, we look at the **tangent space** of G at the identity. Later we will show that this tangent space naturally acquires an algebraic operation — the **Lie bracket** — making it into a **Lie algebra**.

II. Tangent Space T_eG at the Identity

Let G be a smooth manifold and $e \in G$ its identity element.
The **tangent space** at e, denoted T_eG, consists of tangent vectors to all smooth curves through e.

(1) Tangent vector as infinitesimal generator of curve

Each element $X \in T_eG$ corresponds to an **infinitesimal generator** of a smooth curve $g(t) \in G$ with $g(0) = e$:

$$X = \frac{d}{dt} g(t) \Big|_{t=0}.$$

(2) Every tangent vector represents an infinitesimal direction and thus a local subgroup

Every tangent vector $X \in T_e G$ represents an **infinitesimal direction** in the group manifold starting from the identity element e.

To make this precise, pick any **smooth curve** $g(t)$ in G such that:

$$g(0) = e, \qquad \frac{d}{dt} g(t) \Big|_{t=0} = X.$$

Among all such curves, there is a unique one that is also a **group homomorphism** from $(\mathbb{R}, +)$ (near 0) into G:

$$g(t+s) = g(t) g(s), \qquad g(0) = e.$$

Such a curve is called a **local one-parameter subgroup** of G.
Intuitively, it is the path obtained by “flowing” along the group in the direction of X for a time parameter t.

II. Left Translation and Left-Invariant Vector Fields

(1) Left-translation map L_g

In a Lie group G, **left translation** means multiplying every element by a fixed group element g on the **left**:

$$L_g : G \to G, \qquad L_g(h) = gh.$$

This map moves every point $h \in G$ to another point gh along the group’s structure.

(2) Pushforward induced by left-translation

Because L_g is a smooth **map between manifolds**, (in the language of fibre bundles, consider G as the base manifold and (G, TG) a **tangent bundle**), it induces a **pushforward** at each point $h \in G$:

$$(L_g)_* : T_h G \longrightarrow T_{gh} G.$$

That is, $(L_g)_*$ tells us **how a tangent vector at h is carried forward** to a tangent vector at gh under the motion defined by left multiplication.
You can think of $(L_g)_*$ as the infinitesimal version (a differential) of the map L_g — it acts on *velocities* of curves, not on points.

(2') What does $(L_g)_*$ actually do?

Let's recall that a **tangent vector** at a point $h \in G$ can be defined as the **velocity of a smooth curve** through that point:

$$v = \frac{d}{dt}\gamma(t)\bigg|_{t=0}, \qquad \gamma(0) = h,\ \gamma(t) \in G$$

Now apply the **left translation map** $L_g : G \to G$ defined by $L_g(x) = gx$.
Composing the curve with L_g gives a new curve:

$$L_g \circ \gamma : t \mapsto g\gamma(t)$$

which passes through gh at $t = 0$.
The **pushforward** $(L_g)_*$ acts on tangent vectors by differentiating this new curve at $t = 0$:

$$(L_g)_*(v) := \frac{d}{dt}\big(L_g(\gamma(t))\big)\bigg|_{t=0} = \frac{d}{dt}\big(g\gamma(t)\big)\bigg|_{t=0}.$$

> Therefore, $(L_g)_*$ takes a tangent vector v at h — represented by the **motion** $\gamma(t)$ through h — and produces a *new tangent vector* at gh, corresponding to the **same motion**, but now seen through left multiplication by g.

(3) Left-invariant vector field

A **vector field** X on a manifold assigns to each point $h \in G$ a tangent vector $X(h) \in T_hG$. Thus $X : G \to TG$ can be viewed as a smooth map that "attaches an arrow" (a direction in the tangent space) to every point.
A **left-invariant vector field** is a vector field

$$X^L : G \to TG$$

such that it is **invariant** under all left translations:

$$(L_g)_* X^L(h) = X^L(gh), \qquad \forall\, g, h \in G.$$

here "invariant" means as the total manifold is transformed by L_g (a point that is originally h is now gh, the vector attached to this new point is just the pushforward of the vector attached to the original point)

Each tangent vector $X \in T_eG$ uniquely determines a left-invariant vector field X^L by

$$X^L(g) := (L_g)_*X.$$

This establishes a **one-to-one correspondence** between T_eG and the space of left-invariant vector fields on G.

III. Lie Bracket of Vector Fields

(1) Lie braket

Given two smooth vector fields X^L, Y^L on G, their **Lie bracket**

$$[X^L, Y^L] = X^L Y^L - Y^L X^L$$

measures the noncommutativity of their flows — i.e. how performing two infinitesimal displacements in opposite orders differs.

For left-invariant vector fields, $[X^L, Y^L]$ is again left-invariant, so we can define the bracket directly at the identity:

$$[X, Y] := [X^L, Y^L]_e.$$

(2) Lie braket on a Lie group defines a Lie Algebra

A **Lie algebra** over a field $\mathbb{F}$ (typically $\mathbb{R}$ or $\mathbb{C}$) is a vector space $\mathfrak{g}$ equipped with a bilinear map

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

that satisfies:

1. **Antisymmetry:** $[X, Y] = -[Y, X]$
2. **Jacobi identity:** $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Hence, Lie braket defines a bilinear, antisymmetric operation on T_eG satisfying the Jacobi identity. Hence, $(T_eG, [\cdot, \cdot])$ becomes a **Lie algebra**, denoted

$$\mathfrak{g} = \mathrm{Lie}(G).$$

IV. Exponential Map

Now that we have defined left translations and invariant vector fields, we can rigorously introduce the **exponential map**.

For a given $X \in \mathfrak{g} = T_e G$, consider the unique integral curve $g(t)$ of the left-invariant vector field X^L satisfying

$$\frac{d}{dt} g(t) = X^L(g(t)), \qquad g(0) = e.$$

This curve forms a **local one-parameter subgroup** of G:

$$g(t+s) = g(t)g(s)$$

for t, s near 0.

The **exponential map** is then defined by

$$\exp : \mathfrak{g} \to G, \qquad \exp(tX) = g(t)$$

That is, $\exp(X)$ is the group element reached by flowing along X^L for unit parameter $t = 1$.

Interpretations:

- $\exp(tX)$ converts the **infinitesimal generator** X into the **finite transformation** obtained by integrating its flow.
- In matrix Lie groups, the same definition yields the **matrix exponential**:

  $$\exp(X) = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots$$

  and $\exp(tX)$ satisfies $e^{tX}e^{sX} = e^{(t+s)X}$.

Thus, the exponential map provides a precise and geometric bridge between the **Lie algebra** (tangent space of infinitesimal generators) and the **Lie group** (finite symmetry transformations).

VI. Physical Interpretation

In physics:

- The **Lie algebra** corresponds to **infinitesimal symmetry generators** (e.g., angular momentum operators, momentum operators).
- The **Lie group** corresponds to **finite transformations** generated by exponentiation:

$$U(\varepsilon) = e^{-i\varepsilon Q}$$

- The commutator of operators mirrors the Lie bracket:

$$[Q_A, Q_B] = i f_{AB}{}^{C} Q_C$$

  Hence, we say "**Lie algebras describe how infinitesimal symmetry transformations combine**," while "**Lie groups describe how finite transformations act**".

3. Unitary Representations and Generators in QFT

I. Symmetry as a Unitary Representation

In quantum theory, a **symmetry transformation** acts on the Hilbert space of states $\mathcal{H}$ by a **unitary (or anti-unitary)** operator:

$$U(g) : \mathcal{H} \to \mathcal{H}, \qquad g \in G.$$

The map

$$U : G \to \mathrm{U}(\mathcal{H})$$

is called a **unitary representation** of the Lie group if it preserves the group structure:

$$U(g_1 g_2) = U(g_1)\, U(g_2), \qquad U(e) = I$$

(We call it a representation cuz it preserves the group structure, and a unitary one cuz it maps group element to unitary operators)
Thus, each abstract group element g corresponds to a concrete physical operation $U(g)$ acting on states.

II. One-Parameter Unitary Subgroups

Consider a **continuous one-parameter subgroup** of :

$$\{g(\varepsilon) \mid \varepsilon \in \mathbb{R}\}, \qquad g(\varepsilon_1 + \varepsilon_2) = g(\varepsilon_1)g(\varepsilon_2).$$

Its representation on Hilbert space gives a one-parameter family of unitary operators:

$$U(\varepsilon) := U(g(\varepsilon)), \qquad U(0) = I, \quad U(\varepsilon_1 + \varepsilon_2) = U(\varepsilon_1)U(\varepsilon_2).$$

Such a family forms a **strongly continuous one-parameter unitary group**.

III. Stone's Theorem and Self-Adjoint Generators

By **Stone's theorem** (functional analysis):

> Every strongly continuous one-parameter unitary group has a unique **self-adjoint generator** G such that:
>
> $$U(\varepsilon) = e^{-i\varepsilon G},$$
>
> and equivalently,
>
> $$\frac{d}{d\varepsilon}U(\varepsilon)\Big|_{\varepsilon=0} = -iG.$$

- G is called the **generator** of the continuous symmetry.
- Its eigenvalues correspond to **conserved quantities** (e.g., energy, momentum, charge).
- In quantum mechanics, observables are represented by self-adjoint operators, so **generators of unitary symmetries are observables**.

IV. Lie Algebra Representation by Operators

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let

$$U : G \to \mathrm{U}(\mathcal{H})$$

be a **unitary representation** acting on a Hilbert space $\mathcal{H}$.

(1) Infinitesimal form of the representation

For each $X \in \mathfrak{g}$, consider the **one-parameter subgroup**

$$\exp(\varepsilon X) \in G, \qquad \varepsilon \in \mathbb{R}.$$

Its image under the representation is a one-parameter family of unitary operators

$$U(\exp(\varepsilon X)) \equiv U(\varepsilon).$$

By **Stone's theorem**, there exists a unique **self-adjoint (Hermitian) operator** $\hat{X}$ such that

$$U(\exp(\varepsilon X)) = e^{-i\varepsilon \hat{X}}, \qquad \hat{X}^\dagger = \hat{X}.$$

The operator $\hat{X}$ is called the **Hermitian generator** associated with the Lie algebra element X.

(2) Relation between commutators and Lie brackets

Because the exponential map preserves the local group structure,
the product of two infinitesimal transformations satisfies (to first order in ε, δ):

$$\exp(\varepsilon X)\ \exp(\delta Y) = \exp\big(\varepsilon X + \delta Y + \tfrac{1}{2}\varepsilon\delta[X, Y] + \cdots\big).$$

Applying the unitary representation and expanding both sides gives

$$e^{-i\varepsilon \hat{X}} e^{-i\delta \hat{Y}} = e^{-i(\varepsilon \hat{X} + \delta \hat{Y}) - \frac{1}{2}\varepsilon\delta[\hat{X}, \hat{Y}] + \cdots}.$$

Comparing coefficients yields the correspondence between commutators of generators and Lie brackets:

$$[\hat{X}, \hat{Y}] = i\,\widehat{[X, Y]}.$$

of which:

- LHS is the commutator of the unitary representations of Lie algebra elements
- RHS is i times the representation of the Lie braket of two Lie algebra elements

In other words the statement means:

> "The commutator of the **represented generators** on Hilbert space reproduces (up to i) the **representation of the Lie bracket** of the corresponding abstract Lie-algebra elements."

or equivalently,

> "The operator commutator realizes the same algebraic structure as the geometric Lie bracket of vector fields."

(3) Why the factor of i appears

- Mathematically, representations are often taken in **anti-Hermitian** form: $\rho(X)$ satisfies

$$[\rho(X), \rho(Y)] = \rho([X, Y]).$$

- Physicists prefer **Hermitian** generators $\hat{X}$ so that the finite transformations

$$U(\exp(\varepsilon X)) = e^{-i\varepsilon \hat{X}}$$

  are unitary.
- Converting $\rho(X) = -i\hat{X}$ introduces the extra factor of i in

$$[\hat{X}, \hat{Y}] = i\,\widehat{[X, Y]}.$$

(3') Equivalent anti-Hermitian convention

If one defines the **anti-Hermitian representation**

$$\rho(X) := -i\hat{X}$$

then $\rho(X)$ satisfies

$$[\rho(X), \rho(Y)] = \rho([X, Y])$$

which is the mathematicians' convention for Lie-algebra representations.

In this form, $\rho : \mathfrak{g} \to \mathfrak{u}(\mathcal{H})$ is a linear homomorphism into the Lie algebra of anti-Hermitian operators.

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(4) Summary

Mathematical object	Physical meaning	Example
Lie group G	Finite continuous symmetry	Rotations, translations
One-parameter subgroup $\exp(\varepsilon X)$	Continuous transformation parameterized by ε	Rotation by angle ε
Lie algebra element X	Infinitesimal generator (abstract)	Angular momentum axis
Self-adjoint operator $\hat{X}$	Observable generating the symmetry	$\hat{J}_z$ for rotation around z
Exponential $e^{-i\varepsilon\hat{X}}$	Finite unitary transformation	$e^{-i\theta\hat{J}_z}$

For more detailed discussion about the interpretations in QFT, refer to the extension of this notes Lecture 1X Unitary Representations of Lie Group and Lie Algebra in QFT
