Lecture 1X Symmetries of Tensors From Index Manipulation to Representation Theory

0. Motivation

The metric g_{ab} is symmetric;
The electromagnetic tensor $F_{ab} = -F_{ba}$ is antisymmetric;
The Riemann tensor G_{abcd} has "strange" symmetries.

Question: where do these patterns come from?

Are they just conventions, or do they reflect deeper algebraic structure?

Answer: All these symmetries are consequences of how the symmetric group acts on tensor indices

1. Set-up & Notations

- Work over a finite-dimensional vector space V over $\mathbb{R}$ (or $\mathbb{C}$); $\dim V = n$.
- Rank-r tensors live in $V^{\otimes r}$, denotes $T_{a_1 \cdots a_r}$.
  - Rigourously speaking, rank-r tensors live in $V^{*\otimes r}$, but we simplify this for convenience;
  - The notation $T_{\alpha_1 \ldots \alpha_r}$ simply means the r indices of the tensor are labeled by $\alpha_1, \ldots, \alpha_r$; or say, we explicitly choose r symbols for indexing the tensor, and call these symbols $\alpha_1, \ldots, \alpha_r$
- The symmetric group S_r acts by permuting slots:

$$(\sigma \cdot T)_{a_1 \cdots a_r} := T_{a_{\sigma(1)} \cdots a_{\sigma(r)}}.$$

- Round brackets ( ) denote **symmetrization**, and square brackets [ ] denote **antisymmetrization**:

$$T_{(a_1 \cdots a_r)} = \frac{1}{r!} \sum_{\sigma \in S_r} T_{a_{\sigma(1)} \cdots a_{\sigma(r)}}, \quad T_{[a_1 \cdots a_r]} = \frac{1}{r!} \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)\, T_{a_{\sigma(1)} \cdots a_{\sigma(r)}}.$$

2. Rank-2 as Prototype

I. Explicit Decomposition of Rank-2 Tensor

Let T_{ab} be a (covariant) rank-2 tensor.

We can always **decompose** it into:

$$T_{ab} = T_{(ab)} + T_{[ab]}$$

Where:

- $T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba})$ is the **symmetric** part
- $T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba})$ is the **antisymmetric** part

This decomposition is:

- **Linear**: both parts are tensors
- **Unique**: every component of T_{ab} is split unambiguously
- **Complete**: all information in T_{ab} is captured by these two parts

II. Properties of this Decomposition

- The symmetric part satisfies $T_{(ab)} = T_{(ba)}$
- The antisymmetric part satisfies $T_{[ab]} = -T_{[ba]}$
- In n-dimensional space, a rank-2 tensor has:
  - $\frac{n(n+1)}{2}$ independent symmetric components
  - $\frac{n(n-1)}{2}$ independent antisymmetric components
  - So together they span the full n^2-dimensional space of rank-2 tensors.

III. Decomposition of T^{ab} and Symmetry Group S_2

We're interested in how the **permutation of indices** relates to the **representation theory of S_2**.

(1) Permutation group S_2 can be represented by linear automorphism operators on $V \otimes V$

- The group S_2 has two elements:
  - e: identity permutation

- $\sigma = (12)$: swaps $a \leftrightarrow b$
- Now lets consider the **natural action** of the elements, namely their action on space $V^{\otimes 2}$
  - We denote the action of σ on space by $\rho(\sigma)$, and similarly for e by $\rho(e)$
  - The permutation group S_2 acts on this space by permuting tensor **factors**.
    - Recall **tensor factors**.
      - For instance, a pure tensor like $v_1 \otimes v_2 \otimes v_3 \in V^{\otimes 3}$ consists of 3 tensor factors, namely $v_1, v_2, v_3 \in V$.
      - And the space $V^{\otimes 3}$ is made by forming **linear combinations** of such pure tensors
  - S_2 acts on $V^{\otimes 2}$ by permuting tensor factors, for example:
    - (12) acts on $V^{\otimes 2}$ by $\rho(12)$, which permutes tensor factors by: $\rho((12))(v_1 \otimes v_2) = v_2 \otimes v_1$
- We notice that $\rho(\sigma), \rho(e) \in \mathrm{GL}(V^{\otimes 2})$, indicating that the **permutation group** S_2 is **realized by a subgroup of** $\mathrm{GL}(V^{\otimes 2})$, or say be represented by **linear automorphisms** of the tensor product space $V^{\otimes 2}$
  - For general permutation group S_n, we may expect (and indeed, there is) that there exists a group homomorphism:

$$\rho : S_n \to \mathrm{GL}(V^{\otimes n})$$

defined by:

$$\rho(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) := v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

this is called the **natural representation of S_n on $V^{\otimes n}$**

(1‘) Permuation group S_n is naturally represented by linear automorphisms of the tensor product space $V^{\otimes n}$, whose action on tensors is permuting tensor factors

(2) So $V \otimes V$ is a representation space of S_2

Recall that, from representation theory:

> A representation of group G on a vector space W, is a homomorphism:
>
> $$\rho : G \to \mathrm{GL}(W)$$

> i.e. a representation assigns each $g \in G$ a linear automorphism $\rho(g) : W \to W$

So since we now have "S_2 can be represented by (subgroup of) $\mathrm{GL}(V \otimes V)$", we can say that $V \otimes V$ is a **representation space** of the permutation group S_2

(3) Group representation theory tells about relation betwenn subrepresentation, invariant subspace, irreducible representation and projectors

We first recall **Maschke's Theorem**, which states:

> Any finite-dimensional epresentation of G over $\mathbb{C}$ is **completely reducible**:
>
> $$V \cong \bigoplus_{\lambda} m_\lambda V_\lambda$$
>
> where each V_λ carries an irrep, and $m_\lambda \in \mathbb{Z}_{\geq 0}$ is its **multiplicity**;
> equivalently, V is a direct sum of **invariant subspaces**, each **isomorphic to an irrep**; no further invariant splitting is possible inside an irrep.

(And then for each specific irreducible representation labeled λ, the **realization** of a group element $g \in G$ is just an **automorphism operator** $\rho_\lambda(g) \in \mathrm{GL}(V_\lambda)$)

Practically, these invariant subspaces are related to corresponding irreps by the so-called **character (central) projectors**, which is determined by:

$$P_\lambda = \frac{d_\lambda}{|G|} \sum_{g \in G} \chi_\lambda(g^{-1}) \rho(g)$$

(4) Relation between decomposition (of tensor) and projector operator

Idea: the symmetric/antisymmetric piece of rank-3 tensor is exactly the image of the two S_2-equivariant **projectors** built from characters.

1. **Character (central) projectors for S_2.**
   Let χ_+ be the trivial character and χ_- the sign character; $d_+ = d_- = 1$, $|S_2| = 2$
   The projectors can then be determined:

$$P_+ = \frac{1}{2}\big(\rho(e) + \rho(12)\big), \qquad P_- = \frac{1}{2}\big(\rho(e) - \rho(12)\big)$$

These satisfy

$$P_\pm^2 = P_\pm, \quad P_+P_- = 0, \quad P_+ + P_- = I, \quad P_\pm\rho(g) = \rho(g)P_\pm \ \ (\forall g \in S_2)$$

2. **Images = invariant subspaces carrying the two irreps.**

$$\mathrm{Im}\, P_+ = \mathrm{Sym}^2 V$$

(trivial irrep),

$$\mathrm{Im}\, P_- = \Lambda^2 V$$

(sign irrep).
3. Hence the **isotypic decomposition**

$$V \otimes V = \mathrm{Sym}^2 V \oplus \Lambda^2 V$$

is realized by these two commuting idempotents.
4. **Uniqueness, orthogonality, and functoriality.**

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2. More on Group Representation Theory

This section is a short reminder of the core statements of group representation theory.

I. Representation of a Finite Group

A (finite-dimensional) **representation** of a finite group G over $\mathbb{C}$ is a homomorphism

$$\rho : G \to \mathrm{GL}(V)$$

thus we often denote a representation by ρ (the **homomorphism**), or by V (the vector **space** that carries the representations), or (V, ρ) (the representation space along with the homomorphism).

In a specific representation (V, ρ), each group element $g \in G$ is realized by an automorphism operator $\rho(g) \in \mathrm{GL}(V)$ of the space.

II. Invariant Subspace and Irreducible Representation

(1) Invariant subspace

Given a finite-dimensional representation (V, ρ) of group G, we call subspace $W \in V$ a **invariant subspace** if:

$$\forall w \in W, g \in G : \rho(g)w \in W$$

(2) Irreducible representation

(V, ρ) is **irreducible** if it has no nonzero proper invariant subspace.

(3) Schur's Lemma

- **Schur's Lemma:** If V, W are irreps and $T : V \to W$ is G-equivariant, then $T = 0$ unless $V \cong W$; if $V = W$, then $T = \lambda I$.

III. Complete Reducibility (Maschke)

(1) Maschke's Theorem:

For finite G over $\mathbb{C}$, **every representation decomposes as a direct sum of irreps**

$$V \cong \bigoplus_{\lambda} m_\lambda V_\lambda$$

(2) Isotypic decomposition

$$V \cong \bigoplus_{\lambda} V^{(\lambda)}$$

with

$$V^{(\lambda)} \cong V_\lambda \otimes \mathbb{C}^{m_\lambda}$$

Each $V^{(\lambda)}$ is invariant and contains all copies of the irrep V_λ)

(3) character function of an irreducible representation

Let (V, ρ) be a (finite-dimensional, complex) **representation** of a finite group G. Then the **character** of this representation is the complex-valued **function** on G defined by

$$\chi_\rho(g) = \mathrm{Tr}\big(\rho(g)\big)$$

When the representation is **irreducible**, we call this the **character of the irrep**.

IV. Regular Representation

For a finite group G, take the complex vector space

$$\mathbb{C}[G] = \{\sum_{g\in G} a_g, e_g; \Big|; a_g \in \mathbb{C}\}$$

with basis vectors e_g labeled by group elements. Define a **left action** of G by

$$\rho_{\text{reg}}(h), e_g; =; e_{hg} \quad \text{(left multiplication)}$$

This makes $\mathbb{C}[G]$ a $|G|$-dimensional representation space of G; it's called the **(left) regular representation**.

(1) Decomposition of regular representation

One can prove that:

$$\mathbb{C}[G] \cong \bigoplus_\lambda d_\lambda V_\lambda$$

where $d_\lambda = \dim V_\lambda$.

Consequently:

$$\sum_\lambda d_\lambda^2 = |G|$$

characters of irreps are orthonormal (w.r.t. class function inner product).

(2) Decomposition of regular representation gives all irreps of a finite group

V. Projectors (via characters)

- For an irrep λ with character χ_λ and dimension d_λ, the **central idempotent/projector** on the λ-isotypic component is defined as:

$$P_\lambda = \frac{d_\lambda}{|G|} \sum_{g\in G} \chi_\lambda(g^{-1})\rho(g)$$

- Properties:

$$P_\lambda^2 = P_\lambda, \quad P_\lambda P_\mu = 0$$

and

$$\sum_\lambda P_\lambda = I_V$$

VI. Relation with Tensor-Symmetry

- On $V^{\otimes n}$,
  - S_n acts by permuting indices;
  - decomposition into irreps of S_n (Young diagrams) corresponds to symmetry types.
- Young symmetrizers are explicit (non-central) idempotents giving the corresponding subspaces.

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3. Generalization to rank-n tensor and S_n group

I. Natural S_n–action on $V^{\otimes n}$

For a vector space V and $n \geq 2$, define

$$\rho:\ S_n \longrightarrow \mathrm{GL}(V^{\otimes n}), \qquad \rho(\sigma)(v_1 \otimes \cdots \otimes v_n); =; v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

Thus $V^{\otimes n}$ is a representation of S_n (by permuting tensor factors).

II. Schur–Weyl picture (commuting actions)

The actions of S_n and $\mathrm{GL}(V)$ on $V^{\otimes n}$ commute. Consequently there is a canonical decomposition

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n,\ \ell(\lambda) \leq \dim V} S_\lambda(V)\ \otimes\ \mathrm{Specht}_\lambda,$$

where:

- λ is a partition (Young diagram) of n,
- Specht_λ is the irreducible S_n–module of shape λ,
- $S_\lambda(V)$ is the corresponding irreducible $\mathrm{GL}(V)$–module (Schur functor),

- the constraint $\ell(\lambda) \le \dim V$ enforces vanishing components (e.g. $\Lambda^k V = 0$ if $k > \dim V$).

As an S_n–module, the λ–isotypic component is

$$V^{(\lambda)} \cong; \underbrace{\mathrm{Specht}_\lambda}_{S_n \text{ carries}} \otimes \underbrace{S_\lambda(V)}_{\text{multiplicity space}}$$

so the multiplicity of Specht_λ equals $\dim S_\lambda(V)$.

III. Central (character) projectors P_λ

For each irrep λ of S_n with character χ_λ and $d_\lambda = \dim \mathrm{Specht}_\lambda$,

$$P_\lambda = \frac{d_\lambda}{n!} \sum_{\sigma \in S_n} \chi_\lambda(\sigma^{-1}), \rho(\sigma)$$

is a G–equivariant idempotent on $V^{\otimes n}$:

$$P_\lambda^2 = P_\lambda, \quad P_\lambda P_\mu = 0 \ (\lambda \ne \mu), \quad \sum_{\lambda \vdash n} P_\lambda = I, \quad \mathrm{Im}, P_\lambda = V^{(\lambda)}.$$

(1) number of central projectors = number of irreps of S_n = number of partition of n

For any finite group G, the **central primitive idempotents** of $\mathbb{C}[G]$ are in one–to–one correspondence with the **irreducible representations** of G.
For $G = S_n$, this means:

- The number of central primitive idempotents (a.k.a. **central projectors onto isotypic components**) equals the number of **irreps** of S_n.
- The irreps of S_n are classified by **partitions of** n, i.e. by **Young diagrams** of size n.
- Therefore the number is $p(n)$, the number of partitions of n.

(2) How Young diagrams help us find all central projectors?

- Each partition $\lambda \vdash n$ labels:
  - an S_n–irrep Specht_λ (Specht module),
  - a central primitive idempotent $e_\lambda \in \mathbb{C}[S_n]$,
  - and hence a **central projector** P_λ on any S_n–representation via $\rho(P_\lambda)$.

- Concretely, with χ_λ the irreducible character and $d_\lambda = \dim \mathrm{Specht}_\lambda$,

$$e_\lambda = \frac{d_\lambda}{n!} \sum_{\sigma \in S_n} \chi_\lambda(\sigma^{-1}), \sigma \quad (\text{in } \mathbb{C}[S_n])$$

  and for a given action $\rho : S_n \to \mathrm{GL}(V)$,

$$P_\lambda = \rho(e_\lambda)$$

  is the S_n–equivariant projector onto the λ–**isotypic component** of V.

There is another route finding all central projectors via Young diagram:

- Input: a partition / Young diagram
  - Fix a partition $\lambda \vdash n$ (a Young diagram of size n).
  - Choose a **standard Young tableau** t of shape λ (fill $1, \ldots, n$ increasing along rows and columns).
- Row/column subgroups and basic symmetrizers
  - Let $R_t \leq S_n$ be the **row group**: permutations that permute entries **within each row** of t.
  - Let $C_t \leq S_n$ be the **column group**: permutations that permute entries **within each column** of t.
  - Define in the group algebra $\mathbb{C}[S_n]$:

$$a_t \;:=\; \sum_{r \in R_t} r, \qquad b_t \;:=\; \sum_{c \in C_t} \mathrm{sgn}(c)\, c$$

    (Row **symmetrizer** a_t, column **antisymmetrizer** b_t.)
- Young symmetrizer (primitive idempotent up to scale)
  - Define the **Young symmetrizer**:

$$c_t \;:=\; a_t\, b_t \;\in\; \mathbb{C}[S_n].$$

  - There exists a nonzero scalar α_t such that

$$p_t \;:=\; \alpha_t\, c_t \quad \text{satisfies} \quad p_t^2 \;=\; p_t.$$

    (Equivalently, one can take $\alpha_t = 1/f^\lambda$ after choosing a conventional normalization; f^λ is the number of standard Young tableaux of shape λ by the hook–length formula.)
    Meaning: p_t is a **primitive idempotent** in $\mathbb{C}[S_n]$ (not central). Acting via any

representation $\rho : S_n \to \mathrm{GL}(V^{\otimes n})$,

$$\rho(p_t) : V^{\otimes n} \longrightarrow V^{\otimes n}$$

is a projector whose image is **one copy** of the Specht module Specht_λ (with the mixed symmetry encoded by λ).

- Central idempotent (isotypic projector) from summing tableau idempotents
  - Sum the primitive idempotents over **all** standard tableaux t of shape λ:

$$e_\lambda \ := \sum_{t \text{ of shape } \lambda} p_t$$

  - Then e_λ is **central** and **primitive central** in $\mathbb{C}[S_n]$, and its image under any representation ρ is the **λ–isotypic projector**:

$$P_\lambda \ := \ \rho(e_\lambda), \qquad P_\lambda^2 = P_\lambda, \quad P_\lambda P_\mu = 0 \ (\lambda \neq \mu), \quad \sum_{\lambda \vdash n} P_\lambda = I$$

(3) Special cases:

- Totally symmetric part ($\lambda = (n)$):

$$P_{(n)} = \frac{1}{n!} \sum_{\sigma \in S_n} \rho(\sigma) \quad \Rightarrow \quad \mathrm{Im}, P_{(n)} = \mathrm{Sym}^n V$$

- Totally antisymmetric part ($\lambda = (1^n)$):

$$P_{(1^n)} = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma), \rho(\sigma) \quad \Rightarrow \quad \mathrm{Im}, P_{(1^n)} = \Lambda^n V$$

(4) General cases: check (2)

IV. Rank–3 example ($n = 3$)

Partitions: (3) (totally sym), $(2, 1)$ (mixed), (1^3) (totally anti).

- Projectors:

$$P_{(3)} = \tfrac{1}{6} \sum_{\sigma \in S_3} \rho(\sigma), \qquad P_{(1^3)} = \tfrac{1}{6} \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma), \rho(\sigma)$$

and

$$P_{(2,1)} = I - P_{(3)} - P_{(1^3)}$$

- Images:

$$\mathrm{Im}, P_{(3)} = \mathrm{Sym}^3 V, \qquad \mathrm{Im}, P_{(1^3)} = \Lambda^3 V, \qquad \mathrm{Im}, P_{(2,1)} = \text{mixed-symmetry subspace.}$$